

Supplementary Material for: “Dual Principal Component Pursuit: Probability Analysis and Efficient Algorithms”

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1 Basic Notations and Definitions

The Dual Principal Component Pursuit (DPCP) problem refers

$$\min_{\mathbf{b} \in \mathbb{R}^D} f(\mathbf{b}) := \|\tilde{\mathbf{X}}^T \mathbf{b}\|_1 \text{ s.t. } \|\mathbf{b}\|_2 = 1. \quad (1)$$

For any $\mathbf{b} \in \mathbb{R}^D$ and a linear subspace $\mathcal{S}$ with orthonormal projection $\mathcal{P}_{\mathcal{S}}$, the *principal angle* $\phi \in [0, \frac{\pi}{2}]$ of $\mathbf{b}$ from $\mathcal{S}$ is defined as $\cos(\phi) = \|\mathcal{P}_{\mathcal{S}}(\mathbf{b})\|_2 / \|\mathbf{b}\|_2$. We also use $\theta \in [0, \frac{\pi}{2}]$ to denote the principal angle of $\mathbf{b}$ from the normal subspace $\mathcal{S}^\perp$. These two angles satisfy the relationship $\phi + \theta = \frac{\pi}{2}$. Note that $\mathbf{b}$ is a normal vector of $\mathcal{S}$ iff $\theta = 0$ or $\phi = \frac{\pi}{2}$.

We use $\text{sign} : \mathbb{R} \rightarrow \{-1, 0, 1\}$ to denote the sign function as

$$\text{sign}(a) := \begin{cases} a/|a|, & a \neq 0 \\ 0, & a = 0 \end{cases} \quad (2)$$

A related one is the subdifferential Sgn of the absolute function $|a|$:

$$\text{Sgn}(a) := \begin{cases} a/|a|, & a \neq 0 \\ [-1, 1], & a = 0 \end{cases} \quad (3)$$

which is a set. We will also utilize $\text{sgn}(a)$ to represent an arbitrary point in $\text{Sgn}(a)$. Also define

$$c_D := \frac{(D-2)!!}{(D-1)!!} \cdot \begin{cases} \frac{2}{\pi}, & D \text{ is even,} \\ 1, & D \text{ is odd,} \end{cases} \text{ where } k!! := \begin{cases} k(k-2)(k-4) \cdots 4 \cdot 2, & D \text{ is even,} \\ k(k-2)(k-4) \cdots 3 \cdot 1, & D \text{ is odd.} \end{cases} \quad (4)$$

Finally recall several quantities related to the inliers and outliers:

$$c_{\mathbf{X}, \min} := \frac{1}{N} \min_{\mathbf{b} \in \mathcal{S} \cap \mathbb{S}^{D-1}} \|\mathbf{X}^T \mathbf{b}\|_1, \quad c_{\mathbf{X}, \max} := \frac{1}{N} \max_{\mathbf{b} \in \mathcal{S} \cap \mathbb{S}^{D-1}} \|\mathbf{X}^T \mathbf{b}\|_1, \quad (5)$$

$$c_{\mathbf{O}, \min} := \frac{1}{M} \min_{\mathbf{b} \in \mathbb{S}^{D-1}} \|\mathbf{O}^T \mathbf{b}\|_1, \quad c_{\mathbf{O}, \max} := \frac{1}{M} \max_{\mathbf{b} \in \mathbb{S}^{D-1}} \|\mathbf{O}^T \mathbf{b}\|_1, \quad (6)$$

$$\begin{aligned} \eta_{\mathbf{O}} &:= \frac{1}{M} \max_{\mathbf{g}, \mathbf{b} \in \mathbb{S}^{D-1}, \mathbf{g} \perp \mathbf{b}} |\mathbf{g}^T \mathbf{O} \text{sign}(\mathbf{O}^T \mathbf{b})|, & \eta_{\mathbf{X}} &:= \frac{1}{N} \max_{\mathbf{g}, \mathbf{b} \in \mathcal{S} \cap \mathbb{S}^{D-1}, \mathbf{g} \perp \mathbf{b}} |\mathbf{g}^T \mathbf{X} \text{sign}(\mathbf{X}^T \mathbf{b})| \\ \tilde{\eta}_{\mathbf{O}} &:= \frac{1}{M} \max_{\mathbf{b} \in \mathbb{S}^{D-1}} \|(\mathbf{I} - \mathbf{b}\mathbf{b}^T) \mathbf{O} \text{sign}(\mathbf{O}^T \mathbf{b})\|_2, & \tilde{\eta}_{\mathbf{X}} &:= \frac{1}{N} \max_{\mathbf{b} \in \mathcal{S} \cap \mathbb{S}^{D-1}} \|(\mathbf{I} - \mathbf{b}\mathbf{b}^T) \mathbf{X} \text{sign}(\mathbf{X}^T \mathbf{b})\|_2. \\ \bar{\eta}_{\mathbf{O}} &:= \frac{1}{M} \max_{\mathbf{g}, \mathbf{b} \in \mathbb{S}^{D-1}, \mathbf{g} \perp \mathbf{b}} \max_{d \in \text{Sgn}(\mathbf{O}^T \mathbf{b})} |\mathbf{g}^T \mathbf{O} d| \end{aligned} \quad (7)$$

A simple relationship between $\eta_{\mathbf{O}}$ and $\tilde{\eta}_{\mathbf{O}}$ is $\tilde{\eta}_{\mathbf{O}} \leq \sqrt{D-1} \eta_{\mathbf{O}}$. Similarly, we have $\tilde{\eta}_{\mathbf{X}} \leq \sqrt{d-1} \eta_{\mathbf{X}}$. Because of the assumption that the data points are in general position such that every d inliers and every D outliers are linearly independent, we also have $\bar{\eta}_{\mathbf{O}} \leq \eta_{\mathbf{O}} + \min\{M, D\} \leq \eta_{\mathbf{O}} + D$.

2 Proof of Theorem 1

We first repeat Theorem 1.

Theorem 1. Any global solution $\mathbf{b}^*$ to (1) must be orthogonal to the inlier subspace $\mathcal{S}$ as long as

$$\frac{M}{N} \cdot \frac{\sqrt{\bar{\eta}_{\mathcal{O}}^2 + (c_{\mathcal{O},\max} - c_{\mathcal{O},\min})^2}}{c_{\mathcal{X},\min}} < 1. \quad (8)$$

To prove Theorem 1, we need to repeat the following result concerning the critical points.

Lemma 1. Any critical point $\mathbf{b}$ of (1) must either be a normal vector of $\mathcal{S}$, or have a principal angle ϕ from $\mathcal{S}$ smaller than or equal to $\arcsin(M\bar{\eta}_{\mathcal{O}}/Nc_{\mathcal{X},\min})$, where $\bar{\eta}_{\mathcal{O}} := \eta_{\mathcal{O}} + D/M$.

Proof of Theorem 1. Let $\mathbf{b}^*$ be an optimal solution of (1). For the sake of contradiction, suppose that $\mathbf{b}^* \not\perp \mathcal{S}$, i.e., its principal angle $\phi^* < \frac{\pi}{2}$. It follows from Lemma 1 that

$$\sin(\phi^*) \leq \frac{M\bar{\eta}_{\mathcal{O}}}{Nc_{\mathcal{X},\min}}.$$

On the other hand, utilizing the fact that $\mathbf{b}^*$ is a global minimum, we have

$$\cos(\phi^*)Nc_{\mathcal{X},\min} + Mc_{\mathcal{O},\min} \leq g(\mathbf{b}^*) \leq \min_{\mathbf{b} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathcal{S}} \|\tilde{\mathcal{X}}^T \mathbf{b}\|_1 = \min_{\mathbf{b} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathcal{S}} \|\mathcal{O}^T \mathbf{b}\|_1 \leq Mc_{\mathcal{O},\max},$$

which gives

$$\cos(\phi^*) \leq \frac{M(c_{\mathcal{O},\max} - c_{\mathcal{O},\min})}{Nc_{\mathcal{X},\min}}.$$

Combining the above inequalities on ϕ^* yields

$$1 = \sin^2(\phi^*) + \cos^2(\phi^*) \leq \frac{M^2(\bar{\eta}_{\mathcal{O}}^2 + (c_{\mathcal{O},\max} - c_{\mathcal{O},\min})^2)}{N^2 c_{\mathcal{X},\min}^2}.$$

But this contradicts to (8). $\square$

3 Proof of Theorem 2

We first repeat Theorem 2 as follows.

Theorem 2. Consider a random spherical model where the columns of $\mathcal{O}$ and $\mathcal{X}$ are drawn independently and uniformly at random from the unit sphere $\mathbb{S}^{D-1}$ and the intersection of the unit sphere with a subspace $\mathcal{S}$ of dimension $d < D$, respectively. Fix a number $t > 0$. Then with probability at least $1 - 6e^{-t^2/2}$, any global solution of (1) is orthogonal to $\mathcal{S}$ as long as

$$\left((1+t)\sqrt{M} + D\right)^2 + (4+t)^2 cD(\log^2 D)M \leq \left((2/\pi d)N - (2+t/2)\sqrt{N}\right)^2, \quad (9)$$

where c is a universal constant that is independent of N, M, D and d .

The proof of Theorem 2 follows directly from Theorem 1 and the following results concerning different quantities in a random spherical model.

Lemma 2. Consider a random spherical model where the columns of $\mathcal{O}$ and $\mathcal{X}$ are drawn independently and uniformly at random from the unit sphere $\mathbb{S}^{D-1}$ and the intersection of the unit sphere and a subspace $\mathcal{S}$ of dimension $d < D$, respectively. Fix a number $t > 0$. Then with probability at least $1 - 6e^{-t^2/2}$

$$\min \left\{ \begin{array}{l} \mathbb{P}\left[c_{\mathcal{O},\min} \geq c_D - (2 + \frac{t}{2})/\sqrt{M}\right], \mathbb{P}\left[c_{\mathcal{O},\max} \leq c_D + (2 + \frac{t}{2})/\sqrt{M}\right], \\ \mathbb{P}\left[\eta_{\mathcal{O}} \lesssim (1+t)\left(1 + \sqrt{D} \log(\sqrt{c_D D})\right)/\sqrt{M}\right], \mathbb{P}\left[Nc_{\mathcal{X},\min} \geq c_d - (2 + \frac{t}{2})/\sqrt{N}\right], \end{array} \right\} \geq 1 - 2e^{-\frac{t^2}{2}},$$

where $\frac{1}{D} < c_D < 1$ is defined in (4).

In what follows, we prove Lemma 2 by individually converging the four terms $c_{\mathcal{O},\min}$, $c_{\mathcal{O},\max}$, $\eta_{\mathcal{O}}$ and $\eta_{\mathcal{O}}$.

3.1 Preliminaries

Suppose $X_1, \dots, X_n$ are n independent and identically distributed (i.i.d.) random observations from a probability measure P on a measurable space $(\mathcal{X}, \mathcal{A})$. Given a measurable function $f : \mathcal{X} \rightarrow \mathbb{R}$, the *empirical process* evaluated at f is defined as

$$\mathbb{G}_n f := \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n f(X_i) - \int f \, dP \right), \quad (10)$$

where $\int f \, dP$ is the expectation of f under P and $\frac{1}{n} \sum_{i=1}^n f(X_i)$ is called the *empirical distribution*. There are several results concerning the supreme of $\mathbb{G}_n f$ over a given class $\mathcal{F}$ of measurable functions.

Define an *envelope function* $F : \mathcal{X} \rightarrow \mathbb{R}$ such that $|f| \leq F$ for every $f \in \mathcal{F}$. The $L_r(P)$ -norm is defined as $\|f\|_{L_r(P)} = (\int |f|^r \, dP)^{1/r}$. We need one more definition for the so-called *bracket number* which (informally speaking) measures the size of a class functions $\mathcal{F}$. Given two functions l and u , the *bracket* $[l, u]$ is the set of all functions f with $l \leq f \leq u$. An ϵ -bracket in $L_r(P)$ is a bracket $[l, u]$ with $\int (u - l)^r \, dP < \epsilon^r$ (since $l \leq u$, it is equivalent to say $\|u - l\|_{L_r(P)} < \epsilon$). The bracket number $N_{[]}(\epsilon, \mathcal{F}, L_2(P))$ is the minimum number of ϵ -brackets needed to cover $\mathcal{F}$.

Lemma 3 ([1], Cor. 19.35). *For any class $\mathcal{F}$ of measurable functions with envelope function F ,*

$$\mathbb{E} \left[\sup_{f \in \mathcal{F}} |\mathbb{G}_n f| \right] \lesssim J_{[]}(\|F\|_{P,2}, \mathcal{F}, L_2(P)), \quad (11)$$

where $J_{[]}(\|F\|_{P,2}, \mathcal{F}, L_2(P))$ is called the *bracketing integral*:

$$J_{[]}(\|F\|_{L_2(P)}, \mathcal{F}, L_2(P)) = \int_0^{\|F\|_{L_2(P)}} \sqrt{\log(N_{[]}(\epsilon, \mathcal{F}, L_2(P)))} \, d\epsilon. \quad (12)$$

Lemma 4 (McDiarmid's Inequality, [2]). *Let $Z_1, \dots, Z_n$ be real-valued independent random variables. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a function that satisfies*

$$\sup_{z_1, \dots, z_n, z'_i} \left| f(z_1, \dots, z_{i-1}, z_i, z_{i+1}, \dots, z_n) - f(z_1, \dots, z_{i-1}, z'_i, z_{i+1}, \dots, z_n) \right| \leq c_i,$$

for every $i = 1, \dots, n$. Then

$$\mathbb{P} \left[\left| f(Z_1, \dots, Z_n) - \mathbb{E} \left[f(Z_1, \dots, Z_n) \right] \right| \geq \epsilon \right] \leq 2 \exp \left(- \frac{2\epsilon^2}{\sum_{i=1}^n c_i^2} \right).$$

Lemma 5 (Rademacher Comparison, [3], Eqn. (4.20)). *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be convex and increasing. Let $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \leq N$, be 1-Lipschitz functions such that $\varphi_i(0) = 0$. Let ε_i be Rademacher random variables. Then, for any bounded subset T in $\mathbb{R}^N$*

$$\mathbb{E} \left[F \left(\sup_{(t_1, t_2, \dots, t_N) \in T} \sum_{i=1}^N \varepsilon_i \varphi_i(t_i) \right) \right] \leq \mathbb{E} \left[F \left(\sup_{(t_1, t_2, \dots, t_N) \in T} \sum_{i=1}^N \varepsilon_i t_i \right) \right]. \quad (13)$$

Lemma 6 (Rademacher Symmetrization, [4], Thm. 1.1). *Let F be a class of functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that $0 \leq f(z) \leq 1$. Let ε_i be Rademacher random variables. Then for independent and identically distributed random variables $Z_1, \dots, Z_n$, we have*

$$\mathbb{E} \left[\sup_{f \in F} \left(\frac{1}{n} \sum_{i=1}^n f(Z_i) - \mathbb{E}[f(Z)] \right) \right] \leq 2 \mathbb{E} \left[\sup_{f \in F} \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(Z_i) \right], \quad (14)$$

$$\mathbb{E} \left[\sup_{f \in F} \left(\mathbb{E}[f(Z)] - \frac{1}{n} \sum_{i=1}^n f(Z_i) \right) \right] \leq 2 \mathbb{E} \left[\sup_{f \in F} \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(Z_i) \right]. \quad (15)$$

We also require a standard result about the covering number of the sphere. Denote by $\mathcal{N}_\epsilon$ an ϵ -net of $\mathbb{S}^{D-1}$ if every point $\mathbf{b} \in \mathbb{S}^{D-1}$ can be approximated to within ϵ by some point $\mathbf{b}' \in \mathcal{N}_\epsilon$. The minimal cardinality of an ϵ -net, denoted by $\mathcal{N}(\mathbb{S}^{D-1}, \epsilon)$, is called the covering number of $\mathbb{S}^{D-1}$.

Lemma 7 (Covering Number of the Sphere, [5], Lem. 5.2). *For every $\epsilon > 0$, the covering number of the sphere $\mathbb{S}^{D-1}$ satisfies*

$$\mathcal{N}(\mathbb{S}^{D-1}, \epsilon) \leq \left(1 + \frac{2}{\epsilon}\right)^D. \quad (16)$$

We finally require one more result concerning the probability that $\text{sign}(\mathbf{o}^\top \mathbf{b}) = \text{sign}(\mathbf{o}^\top \mathbf{b}')$ when $\mathbf{b}$ is very close to $\mathbf{b}'$.

Lemma 8. *Denote by $\mathbb{B}(\mathbf{b}, \epsilon_1)$ the set of points that around $\mathbf{b}$:*

$$\mathbb{B}(\mathbf{b}, \epsilon_1) := \{\mathbf{b}' \in \mathbb{S}^{D-1} : \|\mathbf{b} - \mathbf{b}'\|_2 \leq \epsilon_1\}.$$

Let $\mathbf{o} \in \mathbb{S}^{D-1}$ be drawn independently and uniformly at random from the unit sphere $\mathbb{S}^{D-1}$. For any $\mathbf{b} \in \mathbb{S}^{D-1}$ and $\epsilon > 0$, define

$$\overline{\mathbb{A}} := \{\mathbf{o} \in \mathbb{S}^{D-1} : \text{sign}(\mathbf{o}^\top \mathbf{b}) = \text{sign}(\mathbf{o}^\top \mathbf{b}'), \forall \mathbf{b}' \in \mathbb{B}(\mathbf{b}, \epsilon_1)\}. \quad (17)$$

Then

$$\mathbb{P}[\mathbf{o} \in \overline{\mathbb{A}}^c] \lesssim \begin{cases} \epsilon, & D = 2, \\ c_D D \epsilon_1^2, & D \geq 3. \end{cases}$$

where $\lesssim$ means smaller than up to a universal constant which is independent of D .

Proof of Lemma 8. Without loss of generality, suppose $\mathbf{b} = \mathbf{e}_1$ which is a length- D vector with 1 in the first entry and 0 elsewhere. When $\epsilon_1 > 1$, note that $\mathbb{P}[\mathbf{o} \in \overline{\mathbb{A}}^c] \leq 1 \lesssim c_D D \epsilon_1^2$. The rest is to consider the more interesting case that $\epsilon_1 \leq 1$. Toward that end, first note that for any $\mathbf{b}' \in \mathbb{B}(\mathbf{b}, \epsilon_1)$, we have $\mathbf{e}_1^\top \mathbf{b}' \leq 1 - \epsilon_1$, which further implies that $\text{sign}(\mathbf{o}^\top \mathbf{b}') = \text{sign}(\mathbf{o}^\top \mathbf{b})$ when $|o_1| \geq \epsilon_1$. Let o_1 be the first element in $\mathbf{o}$, we then have

$$\begin{aligned} \mathbb{P}[\overline{\mathbb{A}}^c] &= \mathbb{P}[\{\mathbf{o} \in \mathbb{S}^{D-1} : \exists \mathbf{b}' \in \mathbb{B}(\mathbf{b}, \epsilon_1), \text{sign}(\mathbf{o}^\top \mathbf{b}) \neq \text{sign}(\mathbf{o}^\top \mathbf{b}')\}] \\ &\leq \mathbb{P}[\{\mathbf{o} \in \mathbb{S}^{D-1} : o_1 \in [-\epsilon_1, \epsilon_1]\}] \end{aligned}$$

To calculate the probability, we use the spherical coordinates. Denote $\mathbf{o}$ as $[\cos(\theta_1), \sin(\theta_1) \cos(\theta_2), \dots, \sin(\theta_1) \cdots \sin(\theta_{D-2}) \cos(\theta_{D-1}), \sin(\theta_1) \cdots \sin(\theta_{D-2}) \sin(\theta_{D-1})]$, where $\theta_1, \dots, \theta_{D-2} \in [0, \pi)$ and $\theta_{D-1} \in [0, 2\pi)$. Also let $I_n = \int_0^\pi \sin^n(\theta) d\theta$. When $D > 2$, let $\mathcal{A}$ be the area of the unit sphere. We have

$$\begin{aligned} \mathbb{P}[\{\mathbf{o} \in \mathbb{S}^{D-1} : o_1 \in [-\epsilon_1, \epsilon_1]\}] &= \frac{\int_{\mathbb{S}^{D-1} \cap o_1 \in [-\epsilon_1, \epsilon_1]} |\cos(\theta_1)| d\mathcal{A}}{\int_{\mathbb{S}^{D-1}} d\mathcal{A}} \\ &= \frac{2 \int_{\arccos(\epsilon_1)}^{\pi/2} \cos(\theta_1) \sin^{D-2}(\theta_1) d\theta_1 \prod_{j=1}^{D-3} I_j}{\prod_{j=1}^{D-2} I_j} = \frac{2 \int_{\arccos(\epsilon_1)}^{\pi/2} \cos(\theta) \sin^{D-2}(\theta) d\theta}{I_{D-2}} \\ &= \frac{2 \sin^{D-1}(\theta) \Big|_{\arccos(\epsilon_1)}^{\pi/2}}{(D-1)I_{D-2}} = 2 \cdot \frac{1 - (1 - \epsilon_1^2)^{(D-1)/2}}{(D-1)I_{D-2}}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{2}{(D-1)I_{D-2}} &= \frac{1}{D-1} \cdot \frac{1}{\int_0^\pi \sin^{D-2}(\theta) d\theta} \\ &= \frac{1}{D-1} \cdot \frac{D-2}{D-3} \cdot \frac{1}{\int_0^\pi \sin^{D-4}(\theta) d\theta} \\ &= \frac{1}{D-1} \cdot \frac{D-2}{D-3} \cdot \frac{D-4}{D-5} \cdots \\ &= c_D. \end{aligned}$$

Then, we apply Taylor's approximation to the term $(1 - \epsilon_1^2)^{(D-1)/2}$ at $\epsilon_1 = 0$.

$$\begin{aligned} (1 - \epsilon_1^2)^{(D-1)/2} &= 1 + 0 \cdot \epsilon_1 + \frac{(D-1)(1-x^2)^{\frac{D-1}{2}}((D-2)x^2-1)}{2(x^2-1)^2} \Big|_0 \epsilon_1^2 + O(\epsilon_1^3) \\ &= 1 - \frac{D-1}{2} \epsilon_1^2 + O(\epsilon_1^3) \\ &\lesssim D\epsilon_1^2. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \mathbb{P}[\{\mathbf{o} \in \mathbb{S}^{D-1} : o_1 \in [-\epsilon_1, \epsilon_1]\}] &= \frac{2}{(D-1)I_{D-2}} \cdot (1 - (1 - \epsilon_1^2)^{(D-1)/2}) \\ &\lesssim c_D D \epsilon_1^2. \end{aligned}$$

The proof of $D = 2$ follows with a similar argument. $\square$

3.2 Bounding $c_{\mathbf{o}, \min}$

We first repeat the result in Lemma 2 concerning $c_{\mathbf{o}, \min}$.

Lemma 9. *Let $\mathbf{o}_1, \dots, \mathbf{o}_M$ be uniformly distributed on $\mathbb{S}^{D-1}$. Then for any $t > 0$*

$$\mathbb{P}\left[c_{\mathbf{o}, \min} \leq c_D - \left(2 + \frac{t}{2}\right) \frac{1}{\sqrt{M}}\right] \leq 2 \exp(-t^2/2).$$

We first present a useful result for proving Lemma 9.

Lemma 10. *Let $\mathbf{o}_1, \dots, \mathbf{o}_M$ be uniformly distributed on $\mathbb{S}^{D-1}$. Then*

$$\begin{aligned} \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \left(\sum_{j=1}^M |\mathbf{b}^T \mathbf{o}_j| - c_D\right)\right] &\leq \frac{2}{\sqrt{M}}, \\ \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \left(c_D - \sum_{j=1}^M |\mathbf{b}^T \mathbf{o}_j|\right)\right] &\leq \frac{2}{\sqrt{M}}. \end{aligned}$$

Proof of Lemma 10. Applying Lemma 6 using the class of functions $F = \{f_{\mathbf{b}} : \mathbb{S}^{D-1} \rightarrow \mathbb{R}; f_{\mathbf{b}}(\mathbf{o}) = |\mathbf{b}^T \mathbf{o}|\}; \mathbf{b} \in \mathbb{S}^{D-1}\}$, we have

$$\begin{aligned} \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \left(\sum_{j=1}^M |\mathbf{b}^T \mathbf{o}_j| - c_D\right)\right] &\leq \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M \varepsilon_j |\mathbf{b}^T \mathbf{o}_j|\right], \\ \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \left(c_D - \sum_{j=1}^M |\mathbf{b}^T \mathbf{o}_j|\right)\right] &\leq \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M \varepsilon_j |\mathbf{b}^T \mathbf{o}_j|\right]. \end{aligned}$$

Next, we apply Lemma 5 with $\varphi_j(\cdot) = |\cdot|$, $t_j = \mathbf{b}^T \mathbf{o}_j$ to get the further bound

$$\begin{aligned} \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M \varepsilon_j |\mathbf{b}^T \mathbf{o}_j|\right] &\leq 2\mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M \varepsilon_j \mathbf{b}^T \mathbf{o}_j\right] = \frac{2}{M} \mathbb{E}\left[\sup_{\|\mathbf{b}\|_2=1} \left\langle \mathbf{b}, \sum_{j=1}^M \varepsilon_j \mathbf{o}_j \right\rangle\right] \\ &= \frac{2}{M} \mathbb{E}\left[\left\|\sum_{j=1}^M \varepsilon_j \mathbf{o}_j\right\|_2\right] \leq \frac{2}{M} \sqrt{\mathbb{E}\left[\left\|\sum_{j=1}^M \varepsilon_j \mathbf{o}_j\right\|_2^2\right]} \\ &= \frac{2}{M} \sqrt{\mathbb{E}\left[M + \sum_{i \neq j} \varepsilon_i \varepsilon_j \mathbf{o}_i^T \mathbf{o}_j\right]} = \frac{2}{M} \sqrt{M} = \frac{2}{\sqrt{M}}, \end{aligned}$$

where the second inequality follows by $\mathbb{E}[Z]^2 \leq \mathbb{E}[Z^2]$. $\square$

We are now ready to prove Lemma 9.

Proof of Lemma 9. First note that

$$Mc_{\mathcal{O},\min} = \inf_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (|\mathbf{b}^T \mathbf{o}_j| - c_D) + Mc_D = Mc_D - \sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (c_D - |\mathbf{b}^T \mathbf{o}_j|).$$

Applying Lemma 10, we have

$$\mathbb{E} \left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (c_D - |\mathbf{b}^T \mathbf{o}_j|) \right] \leq 2\sqrt{M}.$$

Applying Lemma 4 the same way as in proof of Theorem 11, we have

$$\mathbb{P} \left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (c_D - |\mathbf{b}^T \mathbf{o}_j|) \geq 2\sqrt{M} + \epsilon \right] \leq 2 \exp(-2\epsilon^2/M).$$

Therefore,

$$\mathbb{P} \left[Mc_D - \sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (c_D - |\mathbf{b}^T \mathbf{o}_j|) \leq Mc_D - 2\sqrt{M} - \epsilon \right] \leq 2 \exp(-2\epsilon^2/M),$$

and setting $\epsilon = t\sqrt{M}/2$ we get

$$\mathbb{P} \left[Mc_{\mathcal{O},\min} \leq Mc_D - \left(2 + \frac{t}{2}\right) \sqrt{M} \right] \leq 2 \exp(-t^2/2).$$

□

3.3 Bounding $c_{\mathcal{O},\max}$

We first repeat the result in Lemma 2 concerning $c_{\mathcal{O},\max}$.

Lemma 11. *Let $\mathbf{o}_1, \dots, \mathbf{o}_M$ be uniformly distributed on $\mathbb{S}^{D-1}$. Then for any $t > 0$*

$$\mathbb{P} \left[c_{\mathcal{O},\max} \geq c_D + \left(2 + \frac{t}{2}\right) \frac{1}{\sqrt{M}} \right] \leq 2 \exp(-t^2/2).$$

Proof of Lemma 11. First note that

$$Mc_{\mathcal{O},\max} = \sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (|\mathbf{b}^T \mathbf{o}_j| - c_D) + Mc_D.$$

Since $\mathbb{S}^{D-1}$ is compact, there exists $\mathbf{b}^* \in \mathbb{S}^{D-1}$ which achieves the supremum in the expression above. Then for any $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M, \mathbf{v}'_k \in \mathbb{S}^{D-1}$, we have

$$\begin{aligned} & \left| \sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (|\mathbf{b}^T \mathbf{v}_j| - c_D) - \sup_{\|\mathbf{b}\|_2=1} \left(\sum_{j \neq k} (|\mathbf{b}^T \mathbf{v}_j| - c_D) + |\mathbf{b}^T \mathbf{v}'_k| - c_D \right) \right| \\ & \leq \left| \sum_{j=1}^M (|\mathbf{b}^{*T} \mathbf{v}_j| - c_D) - \left(\sum_{j \neq k} (|\mathbf{b}^{*T} \mathbf{v}_j| - c_D) + |\mathbf{b}^{*T} \mathbf{v}'_k| - c_D \right) \right| \\ & = \left| |\mathbf{b}^{*T} \mathbf{v}_k| - |\mathbf{b}^{*T} \mathbf{v}'_k| \right| \leq 1. \end{aligned}$$

Applying Lemma 4 with $c_k = 1$ and using Lemma 10, we obtain

$$\mathbb{P} \left[\sup_{\|\mathbf{b}\|_2=1} \sum_{j=1}^M (|\mathbf{b}^\top \mathbf{o}_j| - c_D) \geq 2\sqrt{M} + \epsilon \right] \leq 2\exp(-2\epsilon^2/M).$$

Finally, set $\epsilon = t\sqrt{M}/2$ to get

$$\mathbb{P} \left[Mc_{\mathbf{o}, \max} \geq Mc_D + \left(2 + \frac{t}{2}\right) \sqrt{M} \right] \leq 2\exp(-t^2/2).$$

□

3.4 Bounding $c_{\mathcal{X}, \min}$

The proof of the result concerning $c_{\mathcal{X}, \min}$ in Lemma 2 follows similar argument as the one for $c_{\mathbf{o}, \min}$ in Section 3.2.

3.5 Bounding $\eta_{\mathbf{o}}$

We first repeat the result in Lemma 2 concerning $\eta_{\mathbf{o}}$.

Lemma 12. *Let $\mathbf{o}_1, \dots, \mathbf{o}_M$ be uniformly distributed on $\mathbb{S}^{D-1}$. Then for any $t > 0$*

$$\mathbb{P} \left[\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right| \gtrsim (1+t)\sqrt{D} \log(\sqrt{c_D D}) \sqrt{M} \right] \leq 2\exp(-t^2/2). \quad (18)$$

Before giving out the main proofs, we first preset the following useful result concerning the expectation of $\eta_{\mathbf{o}}$.

Lemma 13. *Suppose $\mathbf{o}_1, \dots, \mathbf{o}_M$, are drawn independently and uniformly at random from the unit sphere $\mathbb{S}^{D-1}$. Then*

$$\mathbb{E} \left[\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right| \right] \lesssim \sqrt{D} \log(\sqrt{c_D D}) \sqrt{M}, \quad (19)$$

where $\lesssim$ means smaller than up to a universal constant which is independent of D and M .

Proof of Lemma 13. The main idea for proving Lemma 13 is to view

$$\frac{1}{\sqrt{M}} \sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right|$$

as an empirical process and then utilize Lemma 3. Towards that end, define the set

$$\mathbb{F} := \{(\mathbf{b}, \mathbf{g}) : \mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}\}.$$

We further define the parameterized function as

$$f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) := \text{sign}(\mathbf{b}^\top \mathbf{o}) \mathbf{g}^\top \mathbf{o}.$$

The class of functions we are interested in is $\mathcal{F} := \{f_{\mathbf{b}, \mathbf{g}} : (\mathbf{b}, \mathbf{g}) \in \mathbb{F}\}$.

Note that for any $f_{\mathbf{b}, \mathbf{g}} \in \mathcal{F}$ (i.e., $(\mathbf{b}, \mathbf{g}) \in \mathbb{F}$), we have

$$\mathbb{E}[f_{\mathbf{b}, \mathbf{g}}(\mathbf{o})] = \mathbb{E}[\text{sign}(\mathbf{b}^\top \mathbf{o}) \mathbf{g}^\top \mathbf{o}] = 0,$$

which together with (10) indicates that

$$\sum_{j=1}^M \text{sign}(\mathbf{b}^T \mathbf{o}_j) \mathbf{g}^T \mathbf{o}_j = \sqrt{M} \mathbb{G}_M f_{\mathbf{b}, \mathbf{g}},$$

where $\mathbb{G}_M f_{\mathbf{b}, \mathbf{g}}$ is the empirical process of $f_{\mathbf{b}, \mathbf{g}}$.

To utilize Lemma 3, the rest of the proof is to show the corresponding bracketing integral is finite for our problem. Since $|f_{\mathbf{b}, \mathbf{g}}(\mathbf{o})| \leq \|\mathbf{o}\|_2$ for any $(\mathbf{b}, \mathbf{g}) \in \mathbb{F}$, we know $F(\mathbf{o}) = \|\mathbf{o}\|_2$ is the envelope function of $\mathcal{F}$ and $\|F\|_{P, 2} = 1$. Thus, we only need to consider the bracket integral $J_{[]}(\epsilon, \mathcal{F}, L_2(P))$, where P is now a probability measure on the unit sphere. To that end, we first compute the bracket number $N_{[]}(\epsilon, \mathcal{F}, L_2(P))$.

Since our function $f_{\mathbf{b}, \mathbf{g}}$ is parameterized by $(\mathbf{b}, \mathbf{g})$, covering the class of functions $\mathcal{F}$ is related to covering the set $\mathbb{F}$. For any fixed $(\mathbf{b}, \mathbf{g}) \in \mathbb{F}$, define the set of points that around $(\mathbf{b}, \mathbf{g})$:

$$\mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon_1) := \left\{ (\mathbf{b}', \mathbf{g}') \in \mathbb{F} : \sqrt{\|\mathbf{b} - \mathbf{b}'\|_2 + \|\mathbf{g} - \mathbf{g}'\|_2} \leq \epsilon_1 \right\}.$$

Then, denote by

$$\mathbb{A} := \{\mathbf{o} \in \mathbb{R}^D : \text{sign}(\mathbf{o}^T \mathbf{b}) = \text{sign}(\mathbf{o}^T \mathbf{b}'), \forall (\mathbf{b}', \mathbf{g}') \in \mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon_1)\}.$$

When $\mathbf{b}$ is close to $\mathbf{b}'$, then $\mathbb{A}$ should cover most of $\mathbf{o}$. If $\mathbf{o} \in \mathbb{A}$, then for any $(\mathbf{b}', \mathbf{g}') \in \mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon)$ we have

$$|f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) - f_{\mathbf{b}', \mathbf{g}'}(\mathbf{o})| = |\mathbf{o}^T (\mathbf{g} - \mathbf{g}')| \leq \|\mathbf{g} - \mathbf{g}'\|_2 \|\mathbf{o}\|_2 < \epsilon_1.$$

On the other hand, if $\mathbf{o} \in \mathbb{A}^c$, then for any $(\mathbf{b}', \mathbf{g}') \in \mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon_1)$ we have

$$|f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) - f_{\mathbf{b}', \mathbf{g}'}(\mathbf{o})| = |\mathbf{o}^T (\mathbf{g} + \mathbf{g}')| \leq \|\mathbf{g} + \mathbf{g}'\|_2 \|\mathbf{o}\|_2 \leq 2.$$

To summary, we have

$$|f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) - f_{\mathbf{b}', \mathbf{g}'}(\mathbf{o})| \leq \epsilon \delta_{\mathbb{A}}(\mathbf{o}) + 2\delta_{\mathbb{A}^c}(\mathbf{o}), \forall (\mathbf{b}', \mathbf{g}') \in \mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon_1). \quad (20)$$

We now define a bracket $[l, u]$ by

$$\begin{aligned} l(\mathbf{o}) &= f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) - \epsilon_1 \delta_{\mathbb{A}}(\mathbf{o}) - 2\delta_{\mathbb{A}^c}(\mathbf{o}), \\ u(\mathbf{o}) &= f_{\mathbf{b}, \mathbf{g}}(\mathbf{o}) + \epsilon_1 \delta_{\mathbb{A}}(\mathbf{o}) + 2\delta_{\mathbb{A}^c}(\mathbf{o}), \end{aligned}$$

where the indicator function $\delta_{\mathbb{A}}(\mathbf{o})$ is defined as $\delta_{\mathbb{A}}(\mathbf{o}) = \begin{cases} 1, & \mathbf{o} \in \mathbb{A} \\ 0, & \mathbf{o} \in \mathbb{A}^c \end{cases}$. Due to (20), we have $f_{\mathbf{b}', \mathbf{g}'} \in [l, u]$ for all $(\mathbf{b}', \mathbf{g}') \in \mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon_1)$. Also,

$$\begin{aligned} \|u - l\|_{L_2(P)} &= \|2\epsilon \delta_{\mathbb{A}}(\mathbf{o}) + 4\delta_{\mathbb{A}^c}(\mathbf{o})\|_{L_2(P)} = \sqrt{4\epsilon_1^2 \mathbb{P}[\mathbf{o} \in \mathbb{A}] + 16\mathbb{P}[\mathbf{o} \in \mathbb{A}^c]} \\ &< 2\epsilon_1 + 4\sqrt{\mathbb{P}[\mathbf{o} \in \mathbb{A}^c]} \leq 2\epsilon_1 + 4\sqrt{\mathbb{P}[\mathbf{o} \in \bar{\mathbb{A}}^c]} \leq 2\epsilon_1 + 4\sqrt{c_1 c_D D} \epsilon_1, \end{aligned} \quad (21)$$

where $\bar{\mathbb{A}}$ is defined in (17), the second last inequality utilizes the fact $\bar{\mathbb{A}} \subset \mathbb{A}$, last inequality follows because $\mathbb{P}[\mathbf{o} \in \mathbb{A}^c] \leq c_1 c_D D \epsilon_1^2$ according to Lemma 8 (we only consider $D > 2$ here, but the proof for $D = 2$ follows similarly).

Finally, the number of brackets to cover $\mathcal{F}$ is equal to the number of such balls $\mathbb{B}((\mathbf{b}, \mathbf{g}), \epsilon)$ that cover $\mathbb{F}$. Utilizing Lemma 7, the covering number for $\mathbb{F}$ is

$$\mathcal{N}(\mathbb{F}, \epsilon) \leq \left(1 + \frac{2\sqrt{2}}{\epsilon}\right)^{2D}. \quad (22)$$

Recall the definition that the bracket number $N_{[]}(\epsilon, \mathcal{F}, L_2(P))$ is the minimum number of ϵ -brackets needed to cover $\mathcal{F}$, where an ϵ -bracket in $L_2(P)$ is a bracket $[l, u]$ with $\|u - l\|_{L_2(P)} < \epsilon$. Thus, by choosing ϵ_1 such that $2\epsilon_1 + 4\sqrt{c_1 c_D D} \epsilon_1 \leq \epsilon$ and plugging this ϵ_1 into (22), we obtain the bracket number

$$N_{[]}(\epsilon, \mathcal{F}, L_2(P)) \leq \left(1 + c_2 \frac{\sqrt{c_D D}}{\epsilon}\right)^{2D}$$

where c_2 is a universal constant. Now plug this into Lemma 3 gives

$$\frac{1}{\sqrt{M}} \mathbb{E} \left[\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right| \right] \lesssim \int_0^1 \sqrt{\log \left(1 + c_2 \frac{\sqrt{c_D D}}{\epsilon} \right)^{2D}} d\epsilon \lesssim \sqrt{D} \log \left(\sqrt{c_D D} \right).$$

□

We are now ready to prove Lemma 12.

Proof of Lemma 12. Since the product of compact spaces is compact, there exist $\mathbf{b}^*, \mathbf{g}^* \in \mathbb{S}^{D-1}$ for which the value $\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{Sgn}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right|$ is achieved. For $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_M, \mathbf{v}'_k$ any points of $\mathbb{S}^{D-1}$, we have

$$\left| \sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{v}_j) \mathbf{g}^\top \mathbf{v}_j \right| - \sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j \neq k} \text{sign}(\mathbf{b}^\top \mathbf{v}_j) \mathbf{g}^\top \mathbf{v}_j + \text{sgn}(\mathbf{b}^\top \mathbf{v}'_k) \mathbf{g}^\top \mathbf{v}'_k \right| \right| \quad (23)$$

$$\leq \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^{*\top} \mathbf{v}_j) \mathbf{g}^{*\top} \mathbf{v}_j - \sum_{j \neq k} \text{sign}(\mathbf{b}^{*\top} \mathbf{v}_j) \mathbf{g}^{*\top} \mathbf{v}_j + \text{sign}(\mathbf{b}^{*\top} \mathbf{v}'_k) \mathbf{g}^{*\top} \mathbf{v}'_k \right| \quad (24)$$

$$\leq \left| \text{sign}(\mathbf{b}^{*\top} \mathbf{v}_k) \mathbf{g}^{*\top} \mathbf{v}_k - \text{sign}(\mathbf{b}^{*\top} \mathbf{v}'_k) \mathbf{g}^{*\top} \mathbf{v}'_k \right| \leq 2, \quad (25)$$

where the second inequality follows from the reverse triangle inequality. Applying Lemma 4 with $c_k = 2$ and using Lemma 13, we obtain

$$\mathbb{P} \left[\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right| \gtrsim \sqrt{D} \log \left(\sqrt{c_D D} \right) \sqrt{M} \sqrt{M} + \epsilon \right] \leq 2 \exp \left(- \frac{2\epsilon^2}{4M} \right). \quad (26)$$

Finally, set $\epsilon = t\sqrt{M}$ to get

$$\mathbb{P} \left[\sup_{\mathbf{b}, \mathbf{g} \in \mathbb{S}^{D-1}, \mathbf{b} \perp \mathbf{g}} \left| \sum_{j=1}^M \text{sign}(\mathbf{b}^\top \mathbf{o}_j) \mathbf{g}^\top \mathbf{o}_j \right| \gtrsim (1+t) \sqrt{D} \log \left(\sqrt{c_D D} \right) \sqrt{M} \right] \leq 2 \exp(-t^2/2). \quad (27)$$

□

4 Proof of Theorem 3

Algorithm 1 (DPCP-PSGD) Projected Sub-gradient Descent for Solving (1)

Input: data $\tilde{\mathcal{X}} \in \mathbb{R}^{D \times L}$ and initial step size μ_0 ;

Initialization: set $\hat{\mathbf{b}}_0 = \arg \min_{\mathbf{b}} \|\tilde{\mathcal{X}}^\top \mathbf{b}\|_2$, s. t. $\mathbf{b} \in \mathbb{S}^{D-1}$;

1: **for** $k = 1, 2, \dots$ **do**

2: update the step size μ_k according to a certain rule;

3: $\mathbf{b}_k = \hat{\mathbf{b}}_{k-1} - \mu_k \tilde{\mathcal{X}} \text{sign}(\tilde{\mathcal{X}}^\top \hat{\mathbf{b}}_{k-1})$; $\hat{\mathbf{b}}_k = \mathcal{P}_{\mathbb{S}^{D-1}}(\mathbf{b}_k) = \mathbf{b}_k / \|\mathbf{b}_k\|$;

4: **end for**

We first repeat Theorem 3 as follows.

Theorem 3 (Convergence guarantee for PSGD). *Let $\{\hat{\mathbf{b}}_k\}$ be the sequence generated by Algorithm 1 with initialization $\hat{\mathbf{b}}_0$, whose principal angle θ_0 to $\mathcal{S}^\perp$ is assumed to satisfy*

$$\theta_0 < \arctan \left(N c_{\mathcal{X}, \min} / \left(\sqrt{d} N \eta_{\mathcal{X}} + \sqrt{D} M \eta_{\mathcal{O}} \right) \right). \quad (28)$$

Let $\mu' := \frac{1}{4 \cdot \max\{Nc_{\mathcal{X},\min}, Mc_{\mathcal{O},\max}\}}$. Assuming that $Nc_{\mathcal{X},\min} \geq \sqrt{d}N\eta_{\mathcal{X}} + \sqrt{D}M\eta_{\mathcal{O}}$, the angle θ_k between $\widehat{\mathbf{b}}_k$ and $\mathcal{S}^\perp$ satisfies the following properties in accordance with various choices of step sizes.

(i) (constant step size) With $\mu_k = \mu \leq \mu'$, $\forall k \geq 0$, we have

$$\theta_k \leq \begin{cases} \max\{\theta_0, \theta^\diamond(\mu)\}, & k < K^\diamond(\mu), \\ \theta^\diamond(\mu), & k \geq K^\diamond(\mu), \end{cases} \quad (29)$$

where

$$K^\diamond(\mu) := \frac{\tan(\theta_0)}{\mu \left(Nc_{\mathcal{X},\min} - \max\{1, \tan(\theta_0)\} (\sqrt{d}N\eta_{\mathcal{X}} + \sqrt{D}M\eta_{\mathcal{O}}) \right)} \quad (30)$$

and $\theta^\diamond(\mu) := \arctan\left(\frac{\mu}{\sqrt{2}\mu'}\right)$.

(ii) (diminishing step size) With $\mu_k \leq \mu'$, $\mu_k \rightarrow 0$, $\sum_{k=1}^\infty \mu_k = \infty$, we have $\theta_k \rightarrow 0$.

(iii) (diminishing step size of $O(1/k)$) With $\mu_0 \leq \mu'$, $\mu_k = \frac{\mu_0}{k}$, $\forall k \geq 1$, we have $\tan(\theta_k) = O(\frac{1}{k})$.

(iv) (piecewise geometrically diminishing step size) With $\mu_0 \leq \mu'$ and

$$\mu_k = \begin{cases} \mu_0, & k < K_0, \\ \mu_0 \beta^{\lfloor (k-K_0)/K \rfloor + 1}, & k \geq K_0, \end{cases} \quad (31)$$

where $\beta < 1$, $\lfloor \cdot \rfloor$ is the floor function, and $K_0, K \in \mathbb{N}$ are chosen such that

$$K_0 \geq K^\diamond(\mu_0), \quad K \geq \left(\sqrt{2}\beta\mu' \left(Nc_{\mathcal{X},\min} - (\sqrt{d}N\eta_{\mathcal{X}} + \sqrt{D}M\eta_{\mathcal{O}}) \right) \right)^{-1}, \quad (32)$$

where $K^\diamond(\mu)$ is defined in (30), we have

$$\tan(\theta_k) \leq \begin{cases} \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\}, & k < K_0, \\ \frac{\mu_0}{\sqrt{2}\mu'} \beta^{\lfloor (k-K_0)/K \rfloor}, & k \geq K_0. \end{cases} \quad (33)$$

The proof of Theorem 3 builds heavily on the following result.

Lemma 14 (Guarantee for the PSGD). Let $\{\widehat{\mathbf{b}}_k\}$ be the sequence generated by Algorithm 1 is with initialization $\widehat{\mathbf{b}}_0$ whose principal angle to the normal subspace $\mathcal{S}^\perp$ satisfies

$$\theta_0 < \arctan\left(\frac{Nc_{\mathcal{X},\min}}{N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}}\right). \quad (34)$$

and step size satisfying

$$\mu_k \leq \mu' := \frac{1}{4 \cdot \max\{Nc_{\mathcal{X},\min}, c_{\mathcal{O},\max}\}}, \quad \forall k \geq 0. \quad (35)$$

Given

$$c_{\mathcal{X},\min} \geq \tilde{\eta}_{\mathcal{X}} + \frac{M}{N}\tilde{\eta}_{\mathcal{O}}, \quad (36)$$

the angle θ_k of $\widehat{\mathbf{b}}_k$ to $\mathcal{S}^\perp$ satisfies the following properties.

(i) (decay of θ_k when θ_{k-1} is relatively large compared with μ_k) In the case

$$\sin(\theta_{k-1}) > \frac{\mu_k Nc_{\mathcal{X},\min}}{(1 - \mu_k Mc_{\mathcal{O},\max})} \quad (37)$$

we have

$$\tan(\theta_{k-1}) - \tan(\theta_k) \geq \mu_k \min\{Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}), Nc_{\mathcal{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})\}. \quad (38)$$

(ii) (upper bound for θ_k when θ_{k-1} is relatively small compared with μ_k) In the case

$$\sin(\theta_{k-1}) \leq \frac{\mu_k N c_{\mathcal{X}, \min}}{(1 - \mu_k M c_{\mathcal{O}, \max})} \quad (39)$$

we have

$$\begin{aligned} \tan(\theta_k) &\leq \mu_k \max \left\{ \frac{c_{\mathcal{X}, \max} + (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})}{(1 - \mu_k M c_{\mathcal{O}, \max}) \cos(\theta_{k-1})}, \frac{c_{\mathcal{X}, \max}}{(1 - \mu_k M c_{\mathcal{O}, \max}) \cos(\theta_{k-1}) - \mu_k (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})} \right\} \\ &\leq \frac{1}{\sqrt{2}} \frac{\mu_k}{\mu'}. \end{aligned} \quad (40)$$

Before proving Lemma 14, let us first look at why it is possible that PSGD finds a normal vector. To that end, recall that at the k -th step:

$$\mathbf{b}_k = \hat{\mathbf{b}}_{k-1} - \mu_k \tilde{\mathcal{X}} \text{sign}(\tilde{\mathcal{X}}^T \hat{\mathbf{b}}_{k-1}) = \hat{\mathbf{b}}_{k-1} - \mu_k (\mathcal{X} \text{sign}(\mathcal{X}^T \hat{\mathbf{b}}_{k-1}) + \mathcal{O} \text{sign}(\mathcal{O}^T \hat{\mathbf{b}}_{k-1})). \quad (41)$$

Let θ_k be the principal angle of $\hat{\mathbf{b}}_k$ to the orthogonal complement $\mathcal{S}^\perp$. Suppose $\theta_{k-1} > 0$, i.e., $\hat{\mathbf{b}}_{k-1}$ is not a normal vector. We rewrite $\hat{\mathbf{b}}_k$ as $\hat{\mathbf{b}}_k = \sin(\theta_k) \mathbf{s}_k + \cos(\theta_k) \mathbf{n}_k$, where $\mathbf{s}_k \in \mathcal{S}$, $\mathbf{n}_k \perp \mathcal{S}$, $\|\mathbf{s}_k\|_2 = 1$, $\|\mathbf{n}_k\|_2 = 1$. Since $\theta_k > 0$, we have $\text{sign}(\mathcal{X}^T \hat{\mathbf{b}}_{k-1}) = \text{sign}(\mathcal{X}^T \mathbf{s}_{k-1})$. Denote by

$$\begin{aligned} \mathcal{X} \text{sign}(\mathcal{X}^T \hat{\mathbf{b}}_{k-1}) &= \mathcal{X} \text{sign}(\mathcal{X}^T \mathbf{s}_{k-1}) = \mathbf{s}_{k-1} \mathbf{s}_{k-1}^T (\mathcal{X} \text{sign}(\mathcal{X}^T \mathbf{s}_{k-1})) + \underbrace{(\mathcal{P}_{\mathcal{S}} - \mathbf{s}_{k-1} \mathbf{s}_{k-1}^T)}_{\mathbf{p}_{k-1}} (\mathcal{X} \text{sign}(\mathcal{X}^T \mathbf{s}_{k-1})) \\ &= \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1 \mathbf{s}_{k-1} + \mathbf{p}_{k-1}, \\ \mathcal{O} \text{sign}(\mathcal{O}^T \hat{\mathbf{b}}_{k-1}) &= \hat{\mathbf{b}}_{k-1} \hat{\mathbf{b}}_{k-1}^T \mathcal{O} \text{sign}(\mathcal{O}^T \hat{\mathbf{b}}_{k-1}) + \underbrace{(\mathbf{I} - \hat{\mathbf{b}}_{k-1} \hat{\mathbf{b}}_{k-1}^T)}_{\mathbf{q}_{k-1}} \mathcal{O} \text{sign}(\mathcal{O}^T \hat{\mathbf{b}}_{k-1}) \\ &= \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1 \hat{\mathbf{b}}_{k-1} + \mathbf{q}_{k-1}, \end{aligned}$$

where $\mathbf{p}_{k-1}$ and $\mathbf{q}_{k-1}$ are small in the sense that $\|\mathbf{q}_{k-1}\| \leq \tilde{\eta}_{\mathcal{O}}$ and $\|\mathbf{p}_{k-1}\| \leq \tilde{\eta}_{\mathcal{X}}$. Plugging these quantities into (41), we have

$$\begin{aligned} \mathbf{b}_k &= \hat{\mathbf{b}}_{k-1} - \mu_k (\mathcal{X} \text{sign}(\mathcal{X}^T \hat{\mathbf{b}}_{k-1}) + \mathcal{O} \text{sign}(\mathcal{O}^T \hat{\mathbf{b}}_{k-1})) \\ &= \hat{\mathbf{b}}_{k-1} - \mu_k (\|\mathcal{X}^T \mathbf{s}_{k-1}\|_1 \mathbf{s}_{k-1} + \mathbf{p}_{k-1} + \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1 \hat{\mathbf{b}}_{k-1} + \mathbf{q}_{k-1}) \\ &= (1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \hat{\mathbf{b}}_{k-1} - \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1 \mathbf{s}_{k-1} - \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}) \\ &= \left((1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1 \right) \mathbf{s}_{k-1} + (1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) \mathbf{n}_{k-1} \\ &\quad - \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}). \end{aligned} \quad (42)$$

First ignore the last term and assume that θ_{k-1} is not too small such that $(1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) > \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1$. In this case, we know that the principal angle of

$$\left((1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1 \right) \mathbf{s}_{k-1} - (1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) \mathbf{n}_{k-1}$$

is $\arctan \left(\frac{(1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1})} \right)$, which is small than θ_{k-1} . Considering $\mathbf{p}_{k-1}$ and $\mathbf{q}_{k-1}$, the challenge is that we only know their length is small and their direction can be pointed to any position. Thus, when $(1 - \mu_k \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) < \mu_k \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1$, it is difficult (or impossible) to show the decay of the principal angle. For this case, we turn to show that the principal angle will not increase too much.

Proof of Lemma 14. To begin, we first assume that

$$\tan(\theta_k) < \frac{Nc_{\mathbf{x},\min}}{N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}} \quad (43)$$

holds for all $k \geq 0$. As a consequence of (43), we have

$$\begin{aligned} \cos(\theta_k) &= \frac{1}{\sqrt{1 + \tan^2(\theta_k)}} > \frac{1}{\sqrt{1 + \left(\frac{Nc_{\mathbf{x},\min}}{N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}}\right)^2}} \geq \frac{1}{\sqrt{2}} \frac{N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}}{Nc_{\mathbf{x},\min}} \\ &= \frac{\mu'(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})}{\sqrt{2}\mu'Nc_{\mathbf{x},\min}} \geq \frac{\mu'(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})}{(1 - \mu'Mc_{\mathbf{o},\max})}, \end{aligned} \quad (44)$$

where the second inequality follows (36) that $N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o} \leq Nc_{\mathbf{x},\min}$, and the last inequality utilizes (35) that $1 - \mu'Mc_{\mathbf{o},\max} \geq \frac{3}{4} \geq 3\mu'Nc_{\mathbf{x},\min}$. With (43) and (44), we now prove the two main arguments in Lemma 14.

Case (i): We first consider the case where θ_k is large compared with μ_k :

$$\sin(\theta_{k-1}) > \frac{\mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1)}. \quad (45)$$

It follows from (42) that

$$\begin{aligned} \tan(\theta_k) &= \frac{\|\mathcal{P}_S(\mathbf{b}_k)\|_2}{\|\mathcal{P}_{S^\perp}(\mathbf{b}_k)\|_2} \\ &= \frac{\left\| \mathcal{P}_S \left(\left((1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1 \right) \mathbf{s}_{k-1} - \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}) \right) \right\|_2}{\left\| \mathcal{P}_{S^\perp} \left((1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) \mathbf{n}_{k-1} - \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}) \right) \right\|_2} \\ &\leq \frac{\left| (1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) + \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1 \right| + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\ &= \frac{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1 + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\ &= \frac{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1 + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2 + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}. \end{aligned} \quad (46)$$

Based on the term $\Gamma_1 = \frac{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}$, we further bound $\tan(\theta_k)$ from above. In particular, when $\Gamma_1 \leq 1$, we have (utilizing the fact that $\frac{a+x}{b+x}$ is an increasing function of x when $x \geq 0$ and $0 < a \leq b$)

$$\tan(\theta_k) \leq \tan(\bar{\theta}_k) = \frac{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1 + \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1})}$$

by plugging $\|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2 = \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2$ into (46). When $\Gamma_1 > 1$, we have (utilizing the fact that $\frac{a+x}{b+x}$ is a decreasing function of x when $x \geq 0$ and $a > b > 0$)

$$\tan(\theta_k) \leq \tan(\tilde{\theta}_k) = \frac{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{x}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathbf{o}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}$$

by plugging $\|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2 = 0$ into (46). Combining the above two cases together gives

$$\theta_k \leq \max\{\bar{\theta}_k, \tilde{\theta}_k\}. \quad (47)$$

In what follows, we obtain upper bounds for both $\bar{\theta}_k$ and $\tilde{\theta}_k$. Towards that end, we first bound $\tan(\bar{\theta}_k)$ from above as

$$\begin{aligned}
\tan(\bar{\theta}_k) &= \frac{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 + \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1})} \\
&= \tan(\theta_{k-1}) - \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1})} \\
&\leq \tan(\theta_{k-1}) - \mu_k \frac{Nc_{\mathbf{X},\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})}{(1 - \mu_k Mc_{\mathbf{O},\min}) \cos(\theta_{k-1})} \\
&\leq \tan(\theta_{k-1}) - \mu_k \frac{Nc_{\mathbf{X},\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})}{\cos(\theta_{k-1})} \\
&\leq \tan(\theta_{k-1}) - \mu_k (Nc_{\mathbf{X},\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})),
\end{aligned}$$

where the last inequality follows because $0 < 1 - \mu_k c_{\mathbf{O},\min} \leq 1$. Thus,

$$\tan(\theta_{k-1}) - \tan(\bar{\theta}_k) \geq \mu_k (Nc_{\mathbf{X},\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})). \quad (48)$$

Similarly, $\tan(\tilde{\theta}_k)$ can be bounded by

$$\begin{aligned}
\tan(\tilde{\theta}_k) &= \frac{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2} \\
&= \tan(\theta_{k-1}) - \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 - \mu_k \tan(\theta_{k-1}) \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2} \\
&\leq \tan(\theta_{k-1}) - \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 - \mu_k \tan(\theta_{k-1}) \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1})} \\
&\leq \tan(\theta_{k-1}) - \mu_k \frac{c_{\mathbf{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})}{(1 - \mu_k Mc_{\mathbf{O},\min}) \cos(\theta_{k-1})} \\
&\leq \tan(\theta_{k-1}) - \mu_k (Nc_{\mathbf{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})),
\end{aligned}$$

where the forth line utilizes $\|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 \geq c_{\mathbf{X},\min}$, $\|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1 \geq c_{\mathbf{O},\min}$, $\|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2 \leq \|\mathbf{p}_{k-1}\|_2 + \|\mathbf{q}_{k-1}\|_2 \leq N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}$, assumption (43) ensuring $c_{\mathbf{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}) > 0$, and (35) ensuring $1 - \mu_k Mc_{\mathbf{O},\max} > 0$. It follows that

$$\tan(\theta_{k-1}) - \tan(\tilde{\theta}_k) \geq \mu_k (Nc_{\mathbf{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})),$$

which together with (48) and (47) gives

$$\tan(\theta_{k-1}) - \tan(\theta_k) \geq \mu_k \min\{Nc_{\mathbf{X},\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}), Nc_{\mathbf{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})\}. \quad (49)$$

This proves (38).

Case (ii): We now consider the other case where θ_{k-1} is relatively small compared with μ_k :

$$\sin(\theta_{k-1}) \leq \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1)}. \quad (50)$$

In this case, instead of showing that $\theta_k \leq \theta_{k-1}$, we turn to characterize the width of the vibration, i.e., θ_k is also small and if it increase, it will not increase too much. Towards that end, we first bound $\cos(\theta_{k-1})$ as

$$\begin{aligned}
(1 - \mu_k Mc_{\mathbf{O},\max}) \cos(\theta_{k-1}) &= (1 - \mu_k Mc_{\mathbf{O},\max}) \sqrt{1 - \sin^2(\theta_{k-1})} \geq \sqrt{(1 - \mu_k Mc_{\mathbf{O},\max})^2 - (\mu_k Nc_{\mathbf{X},\min})^2} \\
&\geq \sqrt{(1 - \mu_0 Mc_{\mathbf{O},\max})^2 - (\mu_0 Nc_{\mathbf{X},\min})^2} \geq \sqrt{(1 - 1/4)^2 - (1/4)^2} \geq \frac{\sqrt{2}}{2}.
\end{aligned} \quad (51)$$

We use a similar but lightly different approach as in (46) to bound θ_k :

$$\begin{aligned}
\tan(\theta_k) &= \frac{\|\mathcal{P}_S(\mathbf{b}_k)\|_2}{\|\mathcal{P}_{S^\perp}(\mathbf{b}_k)\|_2} = \frac{\left\| \mathcal{P}_S \left(\left((1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 \right) \mathbf{s}_{k-1} + \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}) \right) \right\|_2}{\left\| \mathcal{P}_{S^\perp} \left((1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) \mathbf{n}_{k-1} - \mu_k (\mathbf{p}_{k-1} + \mathbf{q}_{k-1}) \right) \right\|_2} \\
&\leq \frac{\left| (1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) - \mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 \right| + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\
&= \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 - (1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \sin(\theta_{k-1}) + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\
&\leq \frac{\mu_k \|\mathbf{X}^T \mathbf{s}_{k-1}\|_1 + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k \|\mathbf{O}^T \hat{\mathbf{b}}_{k-1}\|_1) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\
&\leq \frac{\mu_k N c_{\mathbf{X}, \max} + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu_k \|\mathcal{P}_{S^\perp}(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2} \\
&= \frac{\mu_k N c_{\mathbf{X}, \max} + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2 + \mu_k \|\mathcal{P}_S(\mathbf{p}_{k-1} + \mathbf{q}_{k-1})\|_2}.
\end{aligned}$$

Similar to the argument utilized in *Case (i)*, and with the abuse of notations as in (47), we have

$$\theta_k \leq \max\{\bar{\theta}_k, \tilde{\theta}_k\},$$

where

$$\begin{aligned}
\tan(\bar{\theta}_k) &= \frac{\mu_k N c_{\mathbf{X}, \max} + \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})} \\
&\leq \frac{\mu_k N c_{\mathbf{X}, \max} + \mu_k (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})} = \mu_k \frac{c_{\mathbf{X}, \max} + (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})},
\end{aligned}$$

and

$$\begin{aligned}
\tan(\tilde{\theta}_k) &= \frac{\mu_k N c_{\mathbf{X}, \max}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu_k \|\mathbf{p}_{k-1} + \mathbf{q}_{k-1}\|_2} \\
&\leq \frac{\mu_k N c_{\mathbf{X}, \min}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu_k (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}.
\end{aligned}$$

The above three equations indicate that

$$\tan(\theta_k) \leq \max \left\{ \mu_k \frac{c_{\mathbf{X}, \max} + (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})}, \frac{\mu_k c_{\mathbf{X}, \max}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu_k (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})} \right\}. \quad (52)$$

The first term in the RHS of (52) can be further bounded by

$$\begin{aligned}
\mu_k \frac{c_{\mathbf{X}, \max} + (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})} &= \frac{\mu_k \mu' c_{\mathbf{X}, \max} + \mu' (N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o})}{\mu' (1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})} \\
&\leq \frac{\mu_k}{\mu'} \frac{\mu' c_{\mathbf{X}, \max} + \mu' N c_{\mathbf{X}, \min}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1})} \leq \frac{1/4 + 1/4}{\sqrt{2}/2} \frac{\mu_k}{\mu'} = \frac{1}{\sqrt{2}} \frac{\mu_k}{\mu'},
\end{aligned}$$

where the first inequality follows from (36) that $N \tilde{\eta} \mathbf{x} + M \tilde{\eta} \mathbf{o} \leq c_{\mathbf{X}, \min}$, and the second inequality we utilize (35) and $(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) \geq \frac{\sqrt{2}}{2}$ from (51). Similarly, the second term in the RHS of (52) can be bounded by

$$\begin{aligned}
\frac{\mu_k N c_{\mathbf{X}, \max}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu' M c_{\mathbf{O}, \max}} &= \frac{\mu_k}{\mu'} \frac{\mu' c_{\mathbf{X}, \max}}{(1 - \mu_k M c_{\mathbf{O}, \max}) \cos(\theta_{k-1}) - \mu' M c_{\mathbf{O}, \max}} \\
&\leq \frac{1/4}{\frac{\sqrt{2}}{2} - 1/4} \frac{\mu_k}{\mu'} = \frac{1}{(2\sqrt{2} - 1)} \frac{\mu_k}{\mu'}.
\end{aligned}$$

where the first inequality follows utilizes (35) and (51). It follows from the above two equations that

$$\begin{aligned} \tan(\theta_k) &\leq \max \left\{ \mu_k \frac{c_{\mathcal{X},\max} + (N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O})}{(1 - \mu_k Mc_{\mathcal{O},\max}) \cos(\theta_{k-1})}, \frac{\mu_k c_{\mathcal{X},\max}}{(1 - \mu_k Mc_{\mathcal{O},\max}) \cos(\theta_{k-1}) - \mu_k (N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O})} \right\} \\ &\leq \frac{1}{\sqrt{2}} \frac{\mu_k}{\mu'}. \end{aligned} \quad (53)$$

Proof of (43): The remaining part is to show that (43) holds for all $k \geq 0$. We prove it by induction. Due to (34), (43) holds for $k = 0$.

In what follows, we suppose that (43) holds for $k \geq 0$ and prove that (43) holds for $k + 1$. Towards that end, we first invoke (49) and (53) to obtain that either $\tan(\theta_{k+1}) < \tan(\theta_k)$ or $\tan(\theta_{k+1}) \leq \frac{1}{\sqrt{2}} \frac{\mu_{k+1}}{\mu'}$. In the former case, we automatically have (43) for $k + 1$ since $\theta_k < \theta_0$.

We now consider the other case $\tan(\theta_{k+1}) \leq \frac{1}{\sqrt{2}} \frac{\mu_{k+1}}{\mu'}$, which along with (35) and (36) gives

$$\tan(\theta_{k+1}) \leq \frac{1}{\sqrt{2}} \frac{\mu_{k+1}}{\mu'} \leq \frac{1}{\sqrt{2}} < \frac{Nc_{\mathcal{X},\min}}{N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O}}.$$

Thus, (43) also holds for $k + 1$. By induction, we conclude that (43) holds for all $k \geq 0$. This completes the proof of Lemma 14. $\square$

In the following four subsections, we prove the four arguments in Theorem 3.

4.1 Proof of Theorem 3(i)

The argument about $k \leq K^\diamond$ follows directly from Lemma 14 which implies that either $\theta_k < \theta_{k-1}$ or θ_k satisfies (29).

We now turn to prove the argument about $k \geq K^\diamond$. First assume that $\theta_{K^\diamond} > 0$ and (37) holds for all $k \leq K^\diamond$, i.e.,

$$\sin(\theta_{k-1}) > \frac{\mu \|\mathcal{X}^T \mathbf{s}_{k-1}\|_1}{(1 - \mu \|\mathcal{O}^T \hat{\mathbf{b}}_{k-1}\|_1)} \geq \frac{\mu Nc_{\mathcal{X},\min}}{(1 - \mu Mc_{\mathcal{O},\min})}, \quad \forall k \leq K^\diamond. \quad (54)$$

It follows from (38) that

$$\tan(\theta_{K^\diamond}) \leq \tan(\theta_0) - K^\diamond \mu (Nc_{\mathcal{X},\min} - \max\{1, \tan(\theta_0)\} (N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O})) \leq 0,$$

which contradicts to the fact that $\theta_{K^\diamond} > 0$. Thus, we have either case (i): $\theta_{K^\diamond} = 0$ or case (ii) there exists $K_1 \leq K^\diamond$ such that

$$\sin(\theta_{K_1-1}) \leq \frac{\mu Nc_{\mathcal{X},\min}}{(1 - \mu c_{\mathcal{O},\max})}. \quad (55)$$

For case (i), due to Lemma 14 which implies that either $\theta_k < \theta_{k-1}$ or (29) holds for all $k \geq K^\diamond + 1$, by induction we conclude that (29) holds for all $k \geq K^\diamond$.

In case (ii), invoking (40), we have

$$\begin{aligned} \tan(\theta_{K_1}) &\leq \mu \max \left\{ \frac{Nc_{\mathcal{X},\max} + (N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O})}{(1 - \mu Mc_{\mathcal{O},\max}) \cos(\theta_{k-1})}, \frac{Nc_{\mathcal{X},\max}}{(1 - \mu c_{\mathcal{O},\max}) \cos(\theta_{k-1}) - \mu (N\tilde{\eta}\mathcal{X} + M\tilde{\eta}\mathcal{O})} \right\} \\ &\leq \frac{1}{\sqrt{2}} \frac{\mu}{\mu'}. \end{aligned}$$

Again Lemma 14 implies that either $\theta_k < \theta_{k-1}$ or (29) holds for all $k \geq K_1 + 1$. By induction, we have that (29) holds for all $k \geq K_1$ and by assumption that $K_1 \leq K^\diamond$. This completes the proof.

4.2 Proof of Theorem 3(ii)

We first show that for any $K_1 \geq 1$, there exists $K_2 \geq K_1$ such that (39) is true for $k = K_2$. We prove it by contradiction. Suppose (37) holds for all $k \geq K_1$, which implies that

$$\tan(\theta_{k-1}) - \tan(\theta_k) \geq \mu_k (Nc_{\mathbf{x},\min} - \max\{1, \tan(\theta_0)\} (N\tilde{\eta}_{\mathbf{x}} + M\tilde{\eta}_{\mathbf{o}})) \quad (56)$$

for all $k \geq K_1$. Repeating the above equation for all $k \geq K_1$ and summing them up give

$$\tan(\theta_{K_1-1}) \geq \tan(\theta_{K_1-1}) - \lim_{k \rightarrow \infty} \tan(\theta_k) \geq (Nc_{\mathbf{x},\min} - (N\tilde{\eta}_{\mathbf{x}} + M\tilde{\eta}_{\mathbf{o}})) \sum_{k=K_1}^{\infty} \mu_k,$$

where we utilize the fact that $\theta_k \in [0, \frac{\pi}{2})$ for all $k \geq 0$. The above equation implies that

$$\sum_{k=K_1}^{\infty} \mu_k < \infty,$$

which contradicts to (31). Thus, there exists $K_2 \geq K_1$ such that (39) is true for $k = K_2$. Now invoking (40), we have

$$\tan(\theta_{K_2}) \leq \frac{1}{\sqrt{2}} \frac{\mu_{K_2}}{\mu'}. \quad (57)$$

For the angle θ_k , $k > K_2$, invoking Lemma 14, we have

$$\theta_k \leq \sup_{i \geq K_2} \frac{1}{\sqrt{2}} \frac{\mu_i}{\mu'}$$

for all $k \geq K_2$. Now letting $K_1 \rightarrow \infty$, which implies that $K_2 \rightarrow \infty$. Then, the above equation along with (31) that $\lim_{i \rightarrow \infty} \mu_i = 0$ gives that

$$\lim_{k \rightarrow \infty} \theta_k \leq 0.$$

Sine $\theta_k \geq 0$, we finally obtain

$$\lim_{k \rightarrow \infty} \theta_k = 0.$$

4.3 Proof of Theorem 3(iii)

It follows from Lemma 14 that at the k -th iteration, we have either

$$\tan(\theta_k) \leq \tan(\theta_{k-1}) - \mu_0 \min\{Nc_{\mathbf{x},\min} - (N\tilde{\eta}_{\mathbf{x}} + M\tilde{\eta}_{\mathbf{o}}), Nc_{\mathbf{x},\min} - \tan(\theta_{k-1})(N\tilde{\eta}_{\mathbf{x}} + M\tilde{\eta}_{\mathbf{o}})\} \frac{1}{k} \quad (58)$$

or

$$\tan(\theta_k) \leq \frac{1}{\sqrt{2}} \frac{\mu_k}{\mu'} = \frac{\mu_0}{\sqrt{2}\mu'} \frac{1}{k}. \quad (59)$$

We first assume that there exists

$$K^* \geq \frac{1}{\sqrt{2}\mu' (Nc_{\mathbf{x},\min} - (N\tilde{\eta}_{\mathbf{x}} + M\tilde{\eta}_{\mathbf{o}}))}$$

such that

$$\tan(\theta_{K^*}) \leq \frac{\mu_0}{\sqrt{2}\mu'} \frac{1}{K^*}. \quad (60)$$

In what follows, we first prove (59) holds for all $k \geq K^*$ by induction. To that end, suppose (59) holds for some $k \geq K^*$, which implies that $\tan(\theta_k) \leq \frac{1}{\sqrt{2}}$. Then we know either (59) holds for $k+1$ or

$$\begin{aligned} \tan(\theta_{k+1}) &\leq \tan(\theta_k) - \mu_0 (Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})) \frac{1}{k+1} \\ &\leq \frac{\mu_0}{\sqrt{2}\mu'} \frac{1}{k} - \mu_0 (Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})) \frac{1}{k+1} \\ &\leq \frac{\mu_0}{\sqrt{2}\mu'} \frac{1}{k+1} - \frac{1}{k+1} \mu_0 \left(c_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}) - \frac{1}{k\sqrt{2}\mu'} \right) \\ &\leq \frac{\mu_0}{\sqrt{2}\mu'} \frac{1}{k+1}, \end{aligned}$$

where the first inequality follows because of (58) and $\tan(\theta_k) \leq \frac{1}{\sqrt{2}}$, and the last line utilizes the fact that $k \geq K^* \geq \frac{1}{\sqrt{2}\mu'(Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}))}$. Thus, (59) holds for all $k \geq K^*$.

The rest of the proof is to show the existence of such K^* . Denote by

$$K_3 = \left\lceil \frac{1}{\sqrt{2}\mu' (Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}))} \right\rceil.$$

By induction, we have

$$\tan(\theta_k) \leq \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\}$$

for all $k \leq K_3$. Now suppose for all $k \geq K_3$, (39) is not true, while implies that (58) must hold. Thus, we have

$$\begin{aligned} \tan(\theta_k) &\leq \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\} - \mu_0 \min\{Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}), Nc_{\mathcal{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})\} \left(\sum_{i=K_3}^k \frac{1}{i} \right) \\ &\leq \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\} - \mu_0 \min\{Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}), Nc_{\mathcal{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})\} \log(k/K_3) \end{aligned}$$

for all $k \geq K_3$. The above equation implies that $\tan(\theta_k) \leq 0$ for all $k \geq K'$ where

$$K' = K_3 e^{\frac{\max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\}}{\mu_0 \min\{Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}), Nc_{\mathcal{X},\min} - \tan(\theta_{k-1})(N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})\}}}.$$

This contradicts to the assumption that (37) always holds. Therefore, there exists at least one $K^* \in [K_3, K']$ such that (60) holds.

For $k < K_3$, since $\tan(\theta_k)$ is always bounded above by $\max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\}$. We have

$$\tan(\theta_k) \leq \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\} K_3 \frac{1}{k}$$

for all $k < K_3$. This together with the fact that (59) holds for all $k \geq K^*$ completes the proof of Theorem 3(iii).

4.4 Proof of Theorem 3(iv)

As illustrated in Figure 1, our main idea is to bound the iterates in each piece or block with Theorem 3(i). To that end, we first use Theorem 3(i) with $\mu = \mu_0$ to get

$$\theta_k \leq \begin{cases} \max\{\theta_0, \theta_0^\diamond\}, & k < K_0^\diamond, \\ \theta_0^\diamond, & k \geq K_0^\diamond, \end{cases} \quad (61)$$

where

$$K_0^\diamond = \left\lceil \frac{\tan(\theta_0)}{\mu_0 \min\{Nc_{\mathcal{X},\min} - (N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}}), c_{\mathcal{X},\min} - \tan(\theta_0)(N\tilde{\eta}_{\mathcal{X}} + M\tilde{\eta}_{\mathcal{O}})\}} \right\rceil \leq K_0 \quad (62)$$

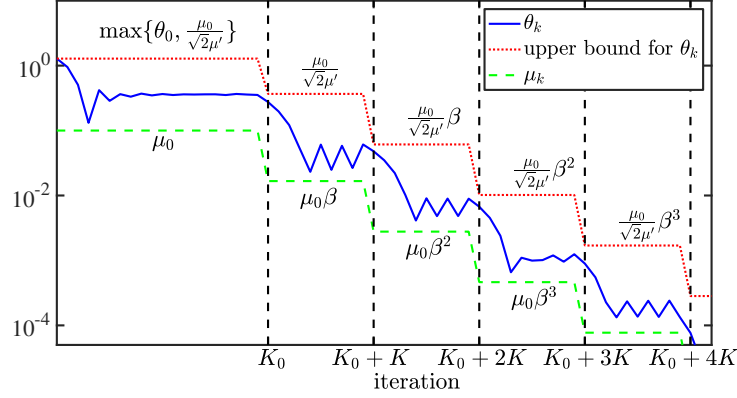


Figure 1: Illustration of (33) guaranteeing the convergence of the PSGD with piecewise exponentially diminishing step size. The red dotted line represents the upper bound in (33), while the green dashed line indicates the choice of piecewise exponentially diminishing step size in (31).

and

$$\tan(\theta_0^\diamond) \leq \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'}. \quad (63)$$

Plugging (62) and (63) back to (61) gives

$$\tan(\theta_k) \leq \begin{cases} \max\{\tan(\theta_0), \frac{\mu_0}{\sqrt{2}\mu'}\}, & k < K_0 \\ \frac{\mu_0}{\sqrt{2}\mu'}, & k \geq K_0. \end{cases}$$

At K_0 -th step, the step size becomes $\beta\mu_0$. We can view it as initialized at $\hat{\mathbf{b}}_{K_0}$ with θ_{K_0} satisfying the above equation. Also, as guaranteed by (43), (34) holds for θ_{K_0} . Thus, applying Theorem 3(i) with $\theta_0 = \theta_{K_0}$ and $\mu = \mu_0\beta$, we have

$$\theta_k \leq \begin{cases} \max\{\theta_{K_0}, \theta_1^\diamond\}, & K_0 \leq k < K_0 + K_1^\diamond, \\ \theta_1^\diamond, & k \geq K_0 + K_1^\diamond, \end{cases}$$

where

$$K_1^\diamond = \left\lceil \frac{\tan(\theta_{K_0})}{\mu_0\beta \min\{Nc\mathbf{x}_{\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}), c\mathbf{x}_{\min} - \tan(\theta_{K_0})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})\}} \right\rceil \quad (64)$$

and

$$\tan(\theta_1^\diamond) \leq \frac{1}{\sqrt{2}} \frac{\mu_0\beta}{\mu'}. \quad (65)$$

It follows from (61)-(63) that

$$\tan(\theta_{K_0}) \leq \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} < 1, \quad (66)$$

which plugged into (64) gives

$$\begin{aligned} K_1^\diamond &= \left\lceil \frac{\tan(\theta_{K_0})}{\mu_0\beta \min\{Nc\mathbf{x}_{\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}), c\mathbf{x}_{\min} - \tan(\theta_{K_0})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})\}} \right\rceil \\ &= \left\lceil \frac{\tan(\theta_{K_0})}{\mu_0\beta (Nc\mathbf{x}_{\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \leq \left\lceil \frac{\frac{1}{\sqrt{2}} \frac{\mu_0\beta}{\mu'}}{\mu_0\beta (Nc\mathbf{x}_{\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \\ &= \left\lceil \frac{1}{\sqrt{2}\beta\mu' (Nc\mathbf{x}_{\min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \leq K. \end{aligned}$$

Plugging the above equation, (65) and (66) into (64) gives

$$\tan(\theta_k) \leq \begin{cases} \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'}, & K_0 \leq k < K + K, \\ \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta, & k \geq K_0 + K. \end{cases} \quad (67)$$

We now complete the proof of (33) by induction. Suppose for some $\ell \geq 1$, the following holds

$$\tan(\theta_k) \leq \begin{cases} \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^{\ell-1}, & K_0 + (\ell-1)K \leq k < K_0 + \ell K, \\ \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^\ell, & k \geq K_0 + \ell K. \end{cases} \quad (68)$$

Similarly, at $(K_0 + \ell K)$ -th step, the step size changes to $\mu_0 \beta^{\ell+1}$. We view it as initialized at $\widehat{\mathbf{b}}_{K_0 + \ell K}$ with $\theta_{K_0 + \ell K}$ satisfying the above equation. Also, as guaranteed by (43), (34) holds for θ_{K_0} . Thus, applying Theorem 3(i) with $\theta_0 = \theta_{K_0 + \ell K}$ and $\mu = \mu_0 \beta^{\ell+1}$, we have

$$\theta_k \leq \begin{cases} \max\{\theta_{K_0}, \theta_{\ell+1}^\circ\}, & K_0 + \ell K \leq k < K_0 + \ell K + K_{\ell+1}^\circ, \\ \theta_{\ell+1}^\circ, & k \geq K_0 + \ell K + K_{\ell+1}^\circ, \end{cases} \quad (69)$$

where

$$K_{\ell+1}^\circ = \left\lceil \frac{\tan(\theta_{K_0 + \ell K})}{\mu_0 \beta^{\ell+1} \min\{Nc_{\mathbf{x}, \min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}), Nc_{\mathbf{x}, \min} - \tan(\theta_{K_0 + \ell K})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})\}} \right\rceil \quad (70)$$

and

$$\tan(\theta_{\ell+1}^\circ) \leq \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^{\ell+1}. \quad (71)$$

Plugging $\theta_{K_0 + \ell K} \leq \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^\ell$ into (69), we have

$$\begin{aligned} K_{\ell+1}^\circ &= \left\lceil \frac{\tan(\theta_{K_0 + \ell K})}{\mu_0 \beta^{\ell+1} \min\{Nc_{\mathbf{x}, \min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}), Nc_{\mathbf{x}, \min} - \tan(\theta_{K_0 + \ell K})(N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o})\}} \right\rceil \\ &= \left\lceil \frac{\tan(\theta_{K_0 + \ell K})}{\mu_0 \beta^{\ell+1} (Nc_{\mathbf{x}, \min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \leq \left\lceil \frac{\frac{1}{\sqrt{2}} \frac{\mu_0 \beta^\ell}{\mu'}}{\mu_0 \beta^{\ell+1} (Nc_{\mathbf{x}, \min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \\ &= \left\lceil \frac{1}{\sqrt{2} \beta \mu' (Nc_{\mathbf{x}, \min} - (N\tilde{\eta}\mathbf{x} + M\tilde{\eta}\mathbf{o}))} \right\rceil \leq K. \end{aligned}$$

Plugging the above equation, (71) and $\theta_{K_0 + \ell K} \leq \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^\ell$ into (69) gives

$$\tan(\theta_k) \leq \begin{cases} \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^{\ell-1}, & K_0 + \ell K \leq k < K_0 + (\ell+1)K, \\ \frac{1}{\sqrt{2}} \frac{\mu_0}{\mu'} \beta^\ell, & k \geq K_0 + (\ell+1)K. \end{cases} \quad (72)$$

By induction, this completes the proof of (33) and hence Theorem 3(iv).

5 Additional Results

5.1 Theoretical Justification for Spectral Initialization

Random Initialization We first consider $\widehat{\mathbf{b}}_0$ drawn randomly from the unit sphere $\mathbb{S}^{D-1}$. For such initialization, we analyze its principal angle to $\mathcal{S}^\perp$ in expectation.

Lemma 15. *Let $\widehat{\mathbf{b}}_0$ be drawn randomly from the unit sphere $\mathbb{S}^{D-1}$. Then*

$$\sqrt{\frac{d-1/2}{D}} \leq \mathbb{E}[\sin(\theta_0)] = \frac{B(\frac{d}{2} + \frac{1}{2}, \frac{D-d}{2})}{B(\frac{d}{2}, \frac{D-d}{2})} \leq \sqrt{\frac{d}{D-1/2}},$$

where θ_0 is the principal angle of $\widehat{\mathbf{b}}_0$ from $\mathcal{S}^\perp$ and $B(\cdot, \cdot)$ is the beta function.

Proof of Lemma 15. It is equivalent to consider a random Gaussian vector $\mathbf{b}_0 \in \mathbb{R}^D$ whose elements are independently generated from a normal distribution. Because the Gaussian distribution is invariant under the orthogonal group of transformations, without loss of generality, we suppose $\mathcal{S} = \text{Span}\{\mathbf{e}_1, \dots, \mathbf{e}_d\}$ where $\mathbf{e}_1, \dots, \mathbf{e}_D$ form a canonical basis of $\mathbb{R}^D$. Now the quantity $\sin^2(\theta_0)$ is simply $\|\mathbf{b}_0(1:d)\|^2/\|\mathbf{b}_0\|^2$ whose distribution function is given by [6, Theorem 3.3.4]:

$$g(z) = \frac{1}{B(\frac{d}{2}, \frac{D-d}{2})} z^{\frac{d}{2}-1} (1-z)^{\frac{D-d}{2}-1},$$

where $B(\cdot, \cdot)$ is the beta function. Hence, we compute the expectation of the quantity $\sin(\theta_0)$

$$\mathbb{E}[\sin(\theta_0)] = \frac{1}{B(\frac{d}{2}, \frac{D-d}{2})} \int_0^1 z^{1/2} z^{\frac{d}{2}-1} (1-z)^{\frac{D-d}{2}-1} dz = \frac{B(\frac{d}{2} + \frac{1}{2}, \frac{D-d}{2})}{B(\frac{d}{2}, \frac{D-d}{2})} = \frac{\gamma(\frac{D}{2})}{\gamma(\frac{D+1}{2})} \frac{\gamma(\frac{d+1}{2})}{\gamma(\frac{d}{2})}.$$

Now plugging the bound $\sqrt{(D-1/2)/2} \leq \frac{\gamma(\frac{D+1}{2})}{\gamma(\frac{D}{2})} \leq \sqrt{D/2}$ into the above equation gives

$$\sqrt{\frac{d-1/2}{D}} \leq \mathbb{E}[\sin(\theta_0)] \leq \sqrt{\frac{d}{D-1/2}}$$

□

Lemma 15 generalizes the result in [7, Lemma 5] which analyzes the inner product between two random vectors on the unit sphere. Lemma 15 indicates that when the subspace dimension d is small, then in expectation a random initialization has a small principal angle from $\mathcal{S}^\perp$. However, if d is large (say $d = D-1$), then it is very possible that a random vector is very close to $\mathcal{S}$, irrespectively of the data matrix. This is because generating a random vector does not utilize the data matrix $\tilde{\mathcal{X}}$, although it is among the easiest ones for initialization. Thus, it is possible to obtain a better initialization from more sophisticated methods using $\tilde{\mathcal{X}}$.

Spectral initialization Another commonly used strategy is to use a spectral method generating an initialization which has much better guaranteed performance than a random one [8]. For our problem, we use the smallest eigenvector of $\tilde{\mathcal{X}}\tilde{\mathcal{X}}^\top$, i.e., the classical PCA approach for finding a normal vector from the data $\tilde{\mathcal{X}}$.

Lemma 16. *Consider a spectral initialization $\mathbf{b}_0$ by taking the smallest eigenvector of $\tilde{\mathcal{X}}\tilde{\mathcal{X}}^\top$. Then, θ_0 (the principal angle of $\mathbf{b}_0$ from $\mathcal{S}^\perp$) satisfies $\sin(\theta_0) < \sqrt{\sigma_1^2(\mathcal{O}) - \sigma_D^2(\mathcal{O})}/\sigma_d(\mathcal{X})$, where σ_ℓ denotes the ℓ -th largest singular value.*

Proof of Lemma 16. Note that for any $\mathbf{b} \perp \mathcal{S}$, $\|\tilde{\mathcal{X}}^\top \mathbf{b}\|^2 = \|\mathcal{O}^\top \mathbf{b}\|^2 \leq \sigma_1^2(\mathcal{O})$. Thus, since $\mathbf{b}_0$ is the optimal solution to $\arg \min_{\mathbf{b} \in \mathbb{S}^{D-1}} \|\tilde{\mathcal{X}}^\top \mathbf{b}\|^2$, we have

$$\|\tilde{\mathcal{X}}^\top \mathbf{b}_0\|^2 \leq \sigma_1^2(\mathcal{O}).$$

On the other hand, we have

$$\|\tilde{\mathcal{X}}^\top \mathbf{b}_0\|^2 = \|\mathcal{X}^\top \mathbf{b}_0\|^2 + \|\tilde{\mathcal{O}}^\top \mathbf{b}_0\|^2 = \sin^2(\theta_0) \|\mathcal{X}^\top \mathbf{s}_0\|^2 + \|\tilde{\mathcal{O}}^\top \mathbf{b}_0\|^2 \geq \sin^2(\theta_0) \sigma_d^2(\mathcal{X}) + \sigma_D^2(\mathcal{O}),$$

where $\sigma_D(\mathcal{O}) = 0$ if $M < D$. Combing the above two equations gives

$$\sin^2(\theta_0) \leq \frac{\sigma_1^2(\mathcal{O}) - \sigma_D^2(\mathcal{O})}{\sigma_d^2(\mathcal{X})}.$$

□

5.2 Improved Analysis of Alternating Linerization and Projection Method

Note that the DPCP problem (1) involves a convex objective function and a nonconvex feasible region, which although nonconvex, is easy to project onto. This inspires us to exploit this structure by solving a sequence of subproblems in which we first solve a convex subproblem with a lineriazation of the nonconvex constraint and then second project onto the sphere. Specifically, if we denote the constraint function associated with (1) as $g(\mathbf{b}) := (\mathbf{b}^T \mathbf{b} - 1)/2$, then the first order Taylor approximation of $g(\mathbf{b})$ at any $\mathbf{w} \in \mathbb{S}^{D-1}$ is $g(\mathbf{b}) \approx g(\mathbf{w}) + \mathbf{b}^T \nabla g(\mathbf{w}) = \mathbf{b}^T \mathbf{w}$. With an initial guess of $\mathbf{b}_0$, we compute a sequence of iterates $\{\mathbf{b}_k\}_{k \geq 1}$ via the update [9]

$$\mathbf{b}_k = \arg \min_{\mathbf{b} \in \mathbb{R}^d} \|\tilde{\mathcal{X}}^T \mathbf{b}\|_1, \text{ s. t. } \mathbf{b}^T \hat{\mathbf{b}}_{k-1} = 1 \quad \text{and} \quad \hat{\mathbf{b}}_k = \mathbf{b}_k / \|\mathbf{b}_k\|_2, \quad (73)$$

where the optimization subproblem can be written as a linear program (LP) rewriting the ℓ_1 norm in an equivalent linear form with auxiliary variables. An alternatively view of the constraint $\mathbf{b}^T \hat{\mathbf{b}}_{k-1} = 1$ in (73) is that it defines an affine hyperplane which excludes the original point $\mathbf{0}$ and has $\hat{\mathbf{b}}_{k-1}$ as its normal vector.

The following result establishes conditions under which $\mathbf{b}_1$ is orthogonal to the subspace $\mathcal{S}$ and contions to guarantee the sequence $\{\hat{\mathbf{b}}_k\}_{k \geq 0}$ converges to a normal vector of $\mathcal{S}$ in a finite number of iterations.

Lemma 17. *Consider the sequence $\{\hat{\mathbf{b}}_k\}_{k \geq 0}$ generated by the recursion (73). Let ϕ_0 be the principal angle of $\hat{\mathbf{b}}_0$ from $\mathcal{S}$. Then,*

(i) $\mathbf{b}_1$ is orthogonal to the subspace $\mathcal{S}$ if

$$\phi_0 > \phi_0^{\natural} := \arctan \left(\frac{Mc_{\mathcal{O}, \max}}{Nc_{\mathcal{X}, \min} - M\bar{\eta}_{\mathcal{O}}} \right); \quad (74)$$

(ii) the sequence $\{\hat{\mathbf{b}}_k\}_{k \geq 0}$ converges to a normal vector of $\mathcal{S}$ in a finite number of iterations if

$$\phi_0 > \phi_0^* := \arccos \left(\frac{\sqrt{N^2 c_{\mathcal{X}, \min}^2 - M^2 \bar{\eta}_{\mathcal{O}}^2} - M(c_{\mathcal{O}, \max} - c_{\mathcal{O}, \min})}{Nc_{\mathcal{X}, \max}} \right). \quad (75)$$

where $c_{\mathcal{X}, \max} := \frac{1}{N} \max_{\mathbf{b} \in \mathcal{S} \cap \mathbb{S}^{D-1}} \|\mathcal{X}^T \mathbf{b}\|_1$.

First note that the expressions in (74) and (75) always define angles between 0 and $\pi/2$ when the condition (8) is satisfied. As illustrated in Figure 2, when the initialization is not very close to the subspace $\mathcal{S}$, Lemma 17 (i) indicates that one procedure of (73) gives a vector that is orthogonal $\mathcal{S}$, i.e., finds a hyperplane that contains all the inliers $\mathcal{X}$. Lemma 17 (ii) suggests that the requirement on the principal angle of the initialization $\hat{\mathbf{b}}_0$ can be weakened if we are allowed to implement multiple alternating procedures in (73). To see this, for well distributed points (both inliers and outliers), the angle ϕ_0^* defined in (75) tends to zero as M, N go to infinity with M/N constant; while $\tan(\phi_0^{\natural})$ defined in (74) goes to $\Omega(M/N)$. We finally note that a similar condition to (75) is also given in [9, Theorem 12]. Compared with the condition in [9, Theorem 12], (75) defines a slightly smaller angle ϕ_0^* and again more importantly, the quantities in (8) is slightly tighter and amenable to probabilistic analysis.

Proof of Lemma 17. We individually prove the two arguments in Lemma 17.

Proof of part (i): For the initialization $\hat{\mathbf{b}}_0$, we first rewrite it by $\hat{\mathbf{b}}_0 = \cos(\phi_0)\mathbf{s}_0 + \sin(\phi_0)\mathbf{n}_0$, where $\mathbf{s}_0 \in \mathcal{S}$, $\mathbf{n}_0 \perp \mathcal{S}$, $\|\mathbf{s}_0\|_2 = 1$, $\|\mathbf{n}_0\|_2 = 1$ and $\phi_0 \in [0, \pi/2]$ is the principal angle of $\mathbf{s}_0$ to the subspace $\mathcal{S}$. For any variable $\mathbf{b}$ in (73) that is not required on the unit sphere, we decompose it by $\mathbf{b} = \alpha\mathbf{s} + \beta\mathbf{n}$, where $\mathbf{s} \in \mathcal{S}$, $\mathbf{n} \perp \mathcal{S}$, $\|\mathbf{s}\|_2 = 1$, $\|\mathbf{n}\|_2 = 1$, $\alpha \geq 0$ and $\beta \geq 0$. Now we rewrite the ℓ_1 minimization in (73) as

$$\begin{aligned} \min_{\alpha, \beta, \mathbf{s}, \mathbf{n}} f_1(\alpha, \beta, \mathbf{s}, \mathbf{n}) &:= \|\tilde{\mathcal{X}}^T(\alpha\mathbf{s} + \beta\mathbf{n})\|_1, \\ \text{s. t. } &\alpha \cos(\phi_0)\mathbf{s}^T \mathbf{s}_0 + \beta \sin(\phi_0)\mathbf{n}^T \mathbf{n}_0 = 1. \end{aligned} \quad (76)$$

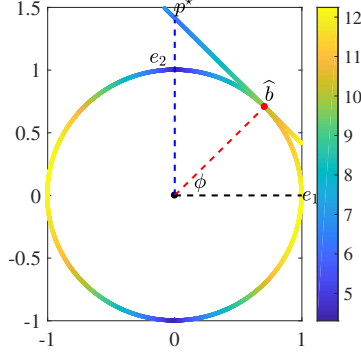


Figure 2: Illustration of Lemma 17 (i). Here $\mathcal{S} = \text{Span}(e_1)$, e_2 is a normal vector and $\hat{\mathbf{b}}_0$ is the initialization for the procedure (73), where the minimization is equivalent to first projecting the circle onto the tangent line passing through $\hat{\mathbf{b}}$ and then finding the smallest re-scaled objective value on this line. .

Note that

$$\|\tilde{\mathcal{X}}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 = \|\mathcal{X}^T \alpha \mathbf{s}\|_1 + \|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1. \quad (77)$$

To show that (76) achieves its global minimum only at $\alpha = 0$ (i.e., $\mathbf{b}$ is a normal vector of $\mathcal{S}$), we first separate α and β in $\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1$ with a surrogate function which is not greater than $\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1$. Specifically, we have

$$\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 \geq \beta \|\mathcal{O}^T \mathbf{n}\|_1 - \alpha \eta_{\mathcal{O}}, \quad (78)$$

where $\hat{\mathbf{b}} = \cos(\phi) \mathbf{s} + \sin(\phi) \mathbf{n}$ is the projection of $\mathbf{b}$ onto the sphere $\mathbb{S}^{D-1}$, i.e., $\phi = \arccos\left(\frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}\right)$. Before proving (78), we note that by triangle inequality of the norm, a simple version of (78) is

$$\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 \geq \beta \|\mathcal{O}^T \mathbf{n}\|_1 - \alpha \|\mathcal{O}^T \mathbf{s}\|_1. \quad (79)$$

However, (79) is too strong in that when we plug (79) into (77), we arrive at $\|\tilde{\mathcal{X}}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 \geq \alpha(\|\mathcal{X}^T \mathbf{s}\|_1 - \|\mathcal{O}^T \mathbf{s}\|_1) + \|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1$, which is useful only when $\|\mathcal{X}^T \mathbf{s}\|_1 - \|\mathcal{O}^T \mathbf{s}\|_1$.

On the other hand, intuitively, when $M \rightarrow \infty$, $\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 \rightarrow c_D \sqrt{\alpha^2 + \beta^2}$ and $\beta \|\mathcal{O}^T \mathbf{n}\|_1 \rightarrow c_D \beta$, which means that $\|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 \geq \beta \|\mathcal{O}^T \mathbf{n}\|_1$ is expected. We now turn to prove (78). To that end, define

$$\chi(\alpha) = \|\mathcal{O}^T(\alpha \mathbf{s} + \beta \mathbf{n})\|_1 - \beta \|\mathcal{O}^T \mathbf{n}\|_1 + \alpha M \bar{\eta}_{\mathcal{O}}. \quad (80)$$

In what follows, we show $\chi(\alpha)$ is an increasing function since together with the fact $\chi(0) = 0$, it is a sufficient condition for (78). Towards that end, we first compute the subdifferential of $\chi(\alpha)$ as

$$\partial \chi(\alpha) = \sum_{i=1}^M \text{Sgn}(\mathbf{o}_i^T(\alpha \mathbf{s} + \beta \mathbf{n})) \mathbf{o}_i^T \mathbf{s} + M \bar{\eta}_{\mathcal{O}} = \sum_{i=1}^M \text{Sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T \mathbf{s} + M \bar{\eta}_{\mathcal{O}}. \quad (81)$$

Now for any $a \in \partial \chi(\alpha)$, we can write it as $a = \sum_{i=1}^M \text{sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T \mathbf{s} + M \bar{\eta}_{\mathcal{O}}$. It follows that

$$\begin{aligned} a &= \sum_{i=1}^M \text{sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T \mathbf{s} + M \bar{\eta}_{\mathcal{O}} \\ &= \sum_{i=1}^M \text{sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T (\cos(\theta)(\cos(\theta) \mathbf{s} + \sin(\theta) \mathbf{n}) - \sin(\theta)(-\sin(\theta) \mathbf{s} + \cos(\theta) \mathbf{n})) + M \bar{\eta}_{\mathcal{O}} \\ &= \cos(\theta) \|\mathcal{O}^T \hat{\mathbf{b}}\|_1 - \sin(\theta) \sum_{i=1}^M \text{sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T \xi_{\hat{\mathbf{b}}} + M \bar{\eta}_{\mathcal{O}} \\ &\geq - \sum_{i=1}^M \text{sgn}(\mathbf{o}_i^T \hat{\mathbf{b}}) \mathbf{o}_i^T \xi_{\hat{\mathbf{b}}} + M \bar{\eta}_{\mathcal{O}} \geq 0 \end{aligned}$$

where $\boldsymbol{\xi}_{\delta} = (-\sin(\theta)\mathbf{s} + \cos(\theta)\mathbf{n})$ and the last inequality follows by the definition of $\bar{\eta}_{\mathcal{O}}$ in (7) (with replacing $\mathbf{b}$ by $\hat{\mathbf{b}}$). Thus, we have $a \geq 0$ for any $a \in \partial\chi(\alpha)$ and therefore (78) follows.

Now plugging (78) into (77), we have

$$\begin{aligned}\|\tilde{\mathcal{X}}^T(\alpha\mathbf{s} + \beta\mathbf{n})\|_1 &= \|\mathcal{X}^T\alpha\mathbf{s}\|_1 + \|\mathcal{O}^T(\alpha\mathbf{s} + \beta\mathbf{n})\|_1 \geq \alpha\|\mathcal{X}^T\mathbf{s}\|_1 + \beta\|\mathcal{O}^T\mathbf{n}\|_1 - \alpha M\bar{\eta}_{\mathcal{O}} \\ &= \alpha(\|\mathcal{X}^T\mathbf{s}\|_1 - M\bar{\eta}_{\mathcal{O}}) + \beta\|\mathcal{O}^T\mathbf{n}\|_1,\end{aligned}$$

where the inequality achieves the equality when $\alpha = 0$. Recalling the assumption that $\|\mathcal{X}^T\mathbf{s}\|_1 - M\bar{\eta}_{\mathcal{O}} > 0$, we now consider the following problem

$$\begin{aligned}\min_{\alpha, \beta, \mathbf{s}, \mathbf{n}} \quad & f_2(\alpha, \beta, \mathbf{s}, \mathbf{n}) := \alpha(\|\mathcal{X}^T\mathbf{s}\|_1 - M\bar{\eta}_{\mathcal{O}}) + \beta\|\mathcal{O}^T\mathbf{n}\|_1, \\ \text{s. t.} \quad & \alpha \cos(\phi_0)\mathbf{s}^T\mathbf{s}_0 + \beta \sin(\phi_0)\mathbf{n}^T\mathbf{n}_0 = 1.\end{aligned}\tag{82}$$

Recall that $f_2(\alpha, \beta, \mathbf{s}, \mathbf{n}) \leq f_1(\alpha, \beta, \mathbf{s}, \mathbf{n})$ and $f_2(\alpha, \beta, \mathbf{s}, \mathbf{n}) = f_1(\alpha, \beta, \mathbf{s}, \mathbf{n})$ when $\alpha = 0$. Thus, if we show that the optimal solution for (82) is obtained only when $\alpha = 0$, we conclude that the optimal solution for (76) is also obtained only when $\alpha = 0$. The remaining part is to consider the global solution of (82).

Suppose that $(\alpha^*, \beta^*, \mathbf{s}^*, \mathbf{n}^*)$ is an optimal solution of (82) with $\alpha^* > 0$. Noting that the l_1 -norm is absolutely scalable, it is clear that $\mathbf{s}_0^T\mathbf{s}^* \geq 0$ and $\mathbf{n}_0^T\mathbf{n}^* \geq 0$. To obtain the contradiction, we construct $\alpha' = 0, \mathbf{s}' = \mathbf{s}^*, \beta'\mathbf{n}' = \beta^*\mathbf{n}^* + \gamma\mathbf{n}_0$ where γ is determined such that $\mathbf{p}' = \beta'\mathbf{n}'$ satisfies the condition $\hat{\mathbf{b}}_0^T\mathbf{p}' = 1$, i.e.,

$$\sin(\phi_0)\beta^*\mathbf{n}_0^T\mathbf{n}^* + \sin(\phi_0)\gamma = 1,$$

which implies that

$$\gamma = \frac{1}{\sin(\phi_0)} - \beta^*\mathbf{n}_0^T\mathbf{n}^*.$$

Since $(\alpha^*, \beta^*, \mathbf{s}^*, \mathbf{n}^*)$ is an optimal solution of (82), it also satisfies the constraint

$$\alpha^* \cos(\phi_0)\mathbf{s}_0^T\mathbf{s}^* + \beta^* \sin(\phi_0)\mathbf{n}_0^T\mathbf{n}^* = 1,$$

which together with the above equation gives

$$\gamma = \frac{1}{\tan(\phi_0)}\alpha^*\mathbf{s}_0^T\mathbf{s}^*.$$

Now we have

$$\begin{aligned}f_2(\alpha^*, \beta^*, \mathbf{s}^*, \mathbf{n}^*) - f_2(\alpha', \beta', \mathbf{s}', \mathbf{n}') &:= \alpha^*(\|\mathcal{X}^T\mathbf{s}^*\|_1 - M\bar{\eta}_{\mathcal{O}}) + \beta^*\|\mathcal{O}^T\mathbf{n}^*\|_1 - \|\mathcal{O}^T(\beta'\mathbf{n}')\|_1 \\ &= \alpha^*(\|\mathcal{X}^T\mathbf{s}^*\|_1 - M\bar{\eta}_{\mathcal{O}}) + \beta^*\|\mathcal{O}^T\mathbf{n}^*\|_1 - \|\mathcal{O}^T(\beta^*\mathbf{n}^* + \gamma\mathbf{n}_0)\|_1 \\ &\geq \alpha^*(\|\mathcal{X}^T\mathbf{s}^*\|_1 - M\bar{\eta}_{\mathcal{O}}) + \beta^*\|\mathcal{O}^T\mathbf{n}^*\|_1 - \beta^*\|\mathcal{O}^T\mathbf{n}^*\|_1 - \gamma\|\mathcal{O}^T\mathbf{n}_0\|_1 \\ &= \alpha^*(\|\mathcal{X}^T\mathbf{s}^*\|_1 - M\bar{\eta}_{\mathcal{O}}) - \frac{1}{\tan(\phi_0)}\alpha^*\mathbf{s}_0^T\mathbf{s}^*\|\mathcal{O}^T\mathbf{n}_0\|_1 \\ &\geq \alpha^*\left(\|\mathcal{X}^T\mathbf{s}^*\|_1 - M\bar{\eta}_{\mathcal{O}} - \frac{1}{\tan(\phi_0)}\|\mathcal{O}^T\mathbf{n}_0\|_1\right) \\ &\geq \alpha^*\left(Nc_{\mathcal{X},\min} - M\bar{\eta}_{\mathcal{O}} - \frac{1}{\tan(\phi_0)}Mc_{\mathcal{O},\max}\right) > 0,\end{aligned}$$

where the last inequality follows because (74) that

$$\phi_0 > \arctan\left(\frac{Mc_{\mathcal{O},\max}}{Nc_{\mathcal{X},\min} - M\bar{\eta}_{\mathcal{O}}}\right).$$

This contradicts to the assumption that $(\alpha^*, \beta^*, \mathbf{s}^*, \mathbf{n}^*)$ is an optimal solution of (82) and thus we conclude that the optimal solution for (82) is obtained only when $\alpha = 0$.

Proof of part (ii): Due to the constraint $\mathbf{b}_k^T \hat{\mathbf{b}}_{k-1} = 1$, we have $\|\mathbf{b}_k\|_2 \geq 1$. It follows that

$$\cdots \leq g(\hat{\mathbf{b}}_k) \leq g(\mathbf{b}_k) \leq g(\hat{\mathbf{b}}_{k-1}) \leq \cdots \leq g(\hat{\mathbf{b}}_0). \quad (83)$$

Invoke [9, Proposition 16] which states that the sequence $\{\mathbf{b}_k\}$ converges to a critical point $\mathbf{b}^*$ of problem (1). For the sake of contradiction, suppose that $\mathbf{b}^* \notin \mathcal{S}$, i.e., its principal angle $\phi^* < \frac{\pi}{2}$. Utilizing (83), we have

$$g(\mathbf{b}^*) = \|\mathcal{X}^T \mathbf{b}^*\|_1 + \|\mathcal{O}^T \mathbf{b}^*\|_1 \leq g(\hat{\mathbf{b}}_0) = \|\mathcal{X}^T \hat{\mathbf{b}}_0\|_1 + \|\mathcal{O}^T \hat{\mathbf{b}}_0\|_1.$$

Plugging the inequalities $\|\mathcal{X}^T \mathbf{b}^*\|_1 \geq N \cos(\phi^*) c_{\mathcal{X}, \min}$, $\|\mathcal{O}^T \mathbf{b}^*\|_1 \geq M c_{\mathcal{O}, \min}$, $\|\mathcal{X}^T \hat{\mathbf{b}}_0\|_1 \leq \cos(\phi_0) N c_{\mathcal{X}, \max}$ and $\|\mathcal{O}^T \hat{\mathbf{b}}_0\|_1 \leq M c_{\mathcal{O}, \max}$ into the above equation gives

$$\cos(\phi^*) \leq \frac{\cos(\phi_0) N c_{\mathcal{X}, \max} + M(c_{\mathcal{O}, \max} - c_{\mathcal{O}, \min})}{N c_{\mathcal{X}, \min}}.$$

On the other hand, since $\mathbf{b}^*$ is a critical point of (1), it follows from the first order optimality (see Lemma 1) that

$$\sin(\phi^*) \leq \frac{M \bar{\eta}_{\mathcal{O}}}{N c_{\mathcal{X}, \min}}.$$

Combining the above equation together gives

$$1 = \cos^2(\phi^*) + \sin^2(\phi^*) \leq \frac{\sqrt{(\cos(\phi_0) N c_{\mathcal{X}, \max} + M(c_{\mathcal{O}, \max} - c_{\mathcal{O}, \min}))^2 + M^2 \bar{\eta}_{\mathcal{O}}^2}}{N c_{\mathcal{X}, \min}},$$

which contradicts (75). □

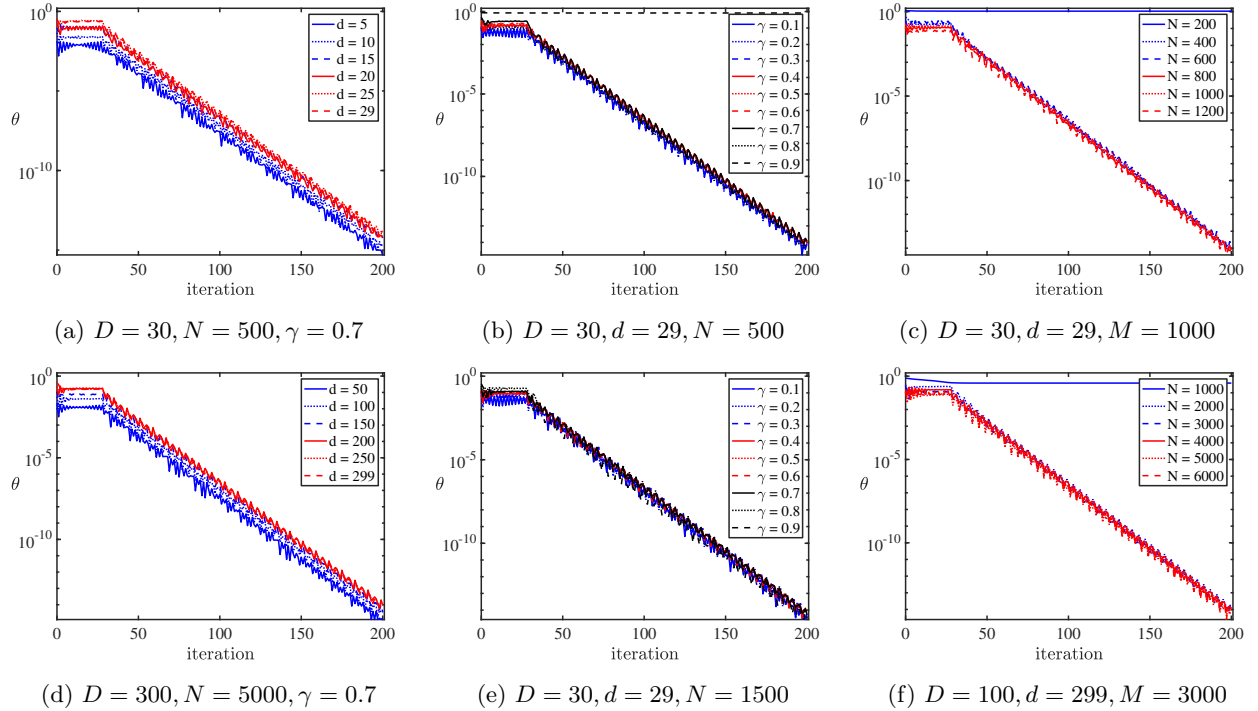


Figure 3: Convergence of the PSGD with piecewise exponentially diminishing step size with $K_0 = 30, K = 4, \beta = \frac{1}{2}$, and μ_0 determined by line search method. Here $\gamma = \frac{M}{M+N}$ denotes the outlier ratio.

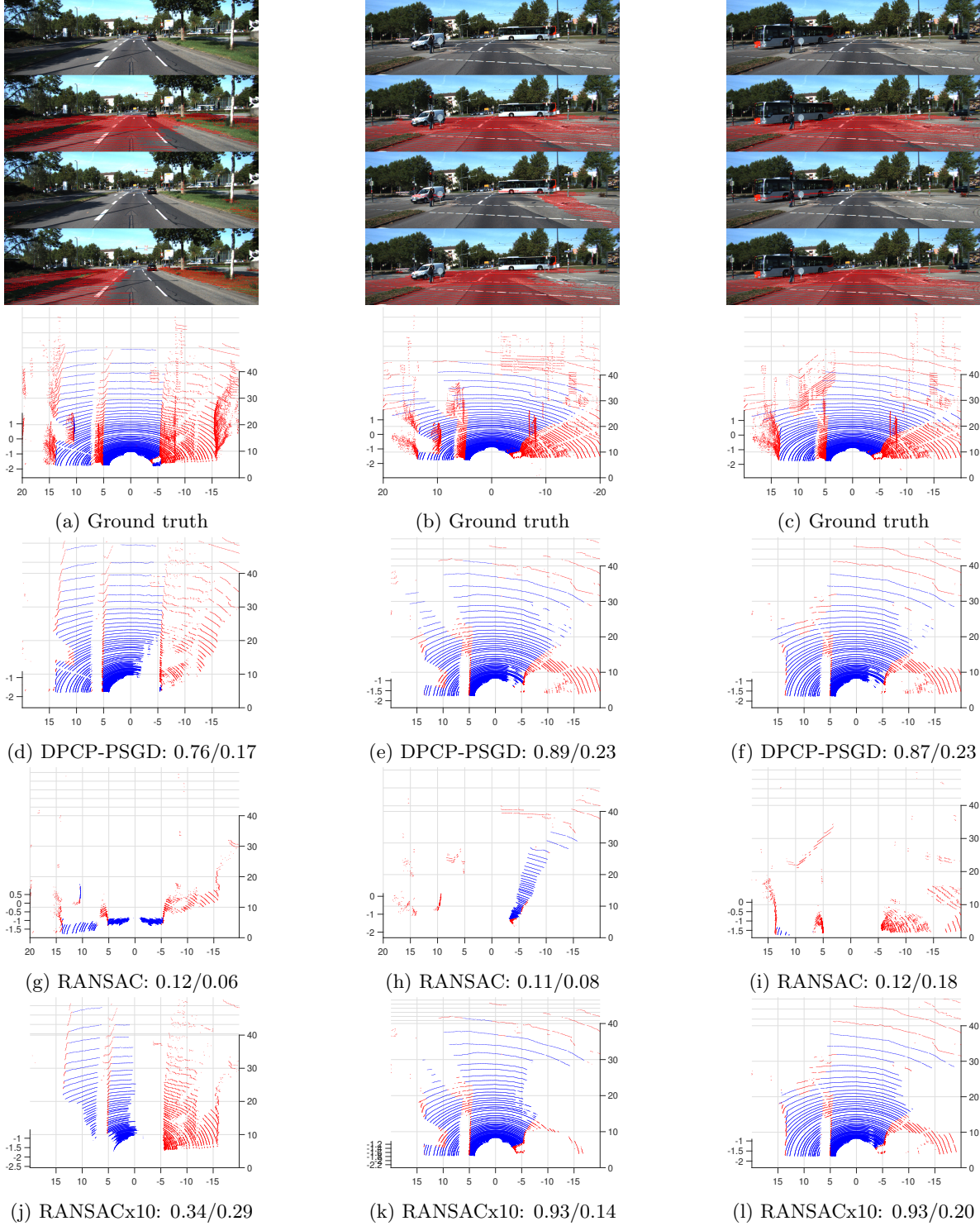


Figure 4: Each column corresponds to a frame. Each image in the first row contains four stacked images: the actual image (top) and the 3D points superimposed on the image that are detected as inliers by DPCP-PSGD (second), by RANSAC using as budget the running time of DPCP-PSGD (third), and by RANSAC using as budget 10 times the running time of DPCP-PSGD (RANSACx10, fourth). In row 2 we depict the ground truth point clouds, with inliers in blue (points on the road plane) and outliers in red (points off the road plane). There are about 100,000 points in each cloud with approximately 46% of them outliers. Rows three, four and five depict the inliers detected by DPCP-PSGD, RANSAC and RANSACx10 respectively, along with the corresponding true-positive and false-positive ratios.

6 Additional Experiments

6.1 Experiments on Synthetical Data for PSGD (Supplementary)

We use synthetic experiments under different settings to further verify the proposed PSGD algorithm with piecewise exponentially diminishing step sizes. We randomly sample a subspace $\mathcal{S}$ of dimension $d < D - 1$, and uniformly at random sample N inliers and M outliers with unit ℓ_2 -norm. Denote by $\gamma = \frac{M}{M+N}$ the outlier ratio. We set $K_0 = 30$ and $K = 4$ for piecewise exponentially diminishing step sizes (as in (32)) with initial step size obtained by one iteration of a backtracking line search. We define $\hat{\mathbf{b}}_0$ to be the bottom eigenvector of $\tilde{\mathbf{x}}\tilde{\mathbf{x}}^T$. Figure 3 displays the convergence of θ (to 0) with different d , D , N and outlier ratio γ . In particular, Figure 3a shows the convergence of θ with $D = 30$, $N = 500$, $\gamma = 0.7$ and different subspace dimension d . We observe R -linear convergence in this case, irrespectively the subspace dimension d . Figure 3d displays similar results but with larger D and N . In Figure 3b, we set $D = 30$, $d = 29$, $N = 500$ and vary the outlier ratio γ from 0.1 to 0.9. We observe R -linear convergence except for the case $\gamma = 0.9$, in which we have much more outliers than inliers. Interestingly, as shown in Figure 3e, when we increase to $N = 1500$ and keep the other parameters the same as in Figure 3b, the PSGD algorithm has R -linear convergence even for $\gamma = 0.9$. This coincides with the fact that the larger N , the more likely the condition (36) is satisfied. Finally we display experiments with varied N in Figure 3c and Figure 3f. We also observe R -linear convergence for PSGD with piecewise exponentially diminishing step sizes given sufficient number of inliers.

6.2 Experiments on Real 3D Point Cloud Road Data (Supplementary)

Here we provide Figure 4, which contains the same results as in the paper except that we include the performance of RANSAC with budget 10 times the running time of DPCP-PSGD. The video on road detection can be found in the link http://nips1105.com/11_0.003.mp4 (where the thresholding parameter is 0.003) and http://nips1105.com/11_0.005.mp4 (where the thresholding parameter is 0.005).

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